Clique-heavy subgraphs and pancyclicity of 2-connected graphs

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Abstract

Graph G on n vertices is said to be pancyclic if it contains cycles of all lengths k for $k \in \{3, ..., n\}$. A vertex $v \in V(G)$ is called super-heavy if the number of its neighbours in G is at least $(n+1)/2$. The complete bipartite graph $K_{1,3}$ is called a claw.

For a given graph H we say that G is H-c_1-heavy if for every induced subgraph K of G isomorphic to H and every maximal clique C in K there is a super-heavy vertex in every non-trivial component of $K - C$; and that G is H-o_1-heavy if in every induced subgraph of G isomorphic to H there are two non-adjacent vertices with sum of degrees at least $|G| + 1$. Let Z_1 denote a graph consisting of a triangle with a pendant edge. In this paper we prove that every 2-connected $K_{1,3}$-o_1-heavy and Z_1-c_1-heavy graph is pancyclic. As a consequence we obtain a complete characterization of claw-o_1-heavy and H-c_1-heavy graphs implying pancyclicity of 2-connected graphs. This result extends previous work by Bedrossian [1].

Keywords: clique-heavy subgraph; Hamilton cycle; Pancyclicity

1. Introduction

We consider only finite, simple and undirected graphs. For terminology and notation not defined here see [4].

Let G be a graph on n vertices. G is said to be hamiltonian, if it contains a cycle C_n, and it is called pancyclic, if it contains cycles of all possible lengths. If G does not contain an induced copy of a given graph H, we say that G is H-free. A vertex $v \in V(G)$ is called heavy (super-heavy) if the

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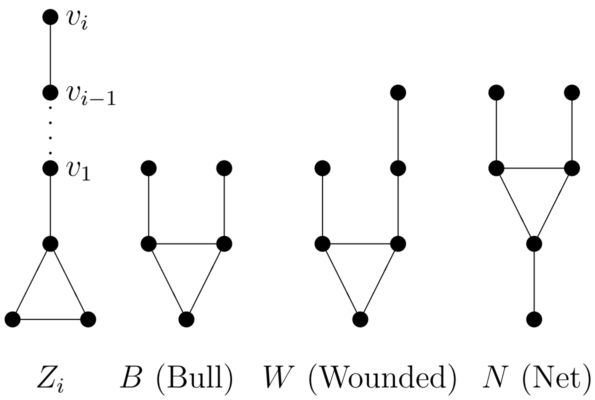

Figure 1: Graphs Z_i, B, W and N

number of its neighbours in G is not less than $n/2$ $((n+1)/2)$. The complete bipartite graph $K_{1,3}$ is called a claw.

One of the most well-known results connecting degree conditions with the existence of hamiltonian cycle in graphs is the following Theorem by Ore.

Theorem 1 (Ore [10]). *Let G be a graph on n vertices. If for every pair of its non-adjacent vertices the sum of their degrees is not less than n, then G is hamiltonian.*

In his PhD thesis Bedrossian considered another approach to the problem of Hamiltonicity, one involving induced subgraphs of 2-connected graphs. He obtained the following interesting result (graphs Z_i, B, W and N are represented on Figure 1).

Theorem 2 (Bedrossian [1]). *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

The "only if" part of this Theorem is due to Faudree and Gould, who presented in [5] infinite families of nonhamiltonian graphs. Forbidding some of the pairs mentioned in the above Theorem turned out to be in fact a sufficient condition for a property stronger than Hamiltonicity.

Theorem 3 (Bedrossian [1]). *Let R and S be connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

In order to relax the conditions of both Theorems it seemed natural to allow these forbidden pairs of subgraphs to be present in a graph but with some degree conditions imposed on them. One of the possibilities is to use Ore-type degree conditions.

Definition 1. We say that an induced subgraph H of a simple graph G is o-heavy (o_1-heavy) in G, if there are two non-adjacent vertices in H with sum of degrees in G at least $|G|$ ($|G| + 1$). Graph G is said to be H-*o*-heavy (H-o_1-heavy) if every induced subgraph of G isomorphic to H is o-heavy (o_1-heavy).

Obviously, every H-free graph is trivially H-o-heavy. Hence the following Theorem extends Bedrossian's result.

Theorem 4 (Li et al. [8])**.** *Let R and S be a connected graphs with R, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-o-heavy implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, C_3, Z_1, Z_2, B, N or W.*

Note that the only pair of subgraphs that appears in Theorem 2 and does not appear here is $\{K_{1,3}, P_6\}$. The authors of the above Theorem present in [8] non-hamiltonian $\{K_{1,3}, P_6\}$-o-heavy (and even claw-free and P_6-o-heavy) graphs. Again, the same pairs as in the case of forbidden subgraphs provide in fact a sufficient condition for pancyclicity, with a slightly stronger requirement for the sums of degrees of non-adjacent vertices.

Theorem 5 (Li et al. [7])**.** *Let G be a 2-connected graph which is not a cycle and let R and S be connected graphs with R, $S \neq P_3$. Then G being $\{R, S\}$-o_1-heavy implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4$, P_5, Z_1, Z_2.*

Recently ([6]) Li and Ning introduced another type of heavy graphs.

Definition 2. We say that an induced subgraph H of a simple graph G is c-heavy (c_1-heavy) in G, if for every maximal clique C of H every non-trivial component of $H - C$ contains a vertex that is heavy in G. Graph G is said to be H-*c*-heavy (H-c_1-heavy) if every induced subgraph of G isomorphic to H is c-heavy (c_1-heavy).

Note that, in general case, properties of being c-heavy and o-heavy are independent, in the sense that none of them implies another. Furthermore, there is no point in examining claw-c-heavy graphs, as the notion is in this case meaningless (every component of the claw lacking maximal clique is trivial). Thus the authors considered pairs of claw-o-heavy and S-c-heavy graphs and obtained the following result.

Theorem 6 (Li, Ning [6]). *Let S be a connected graph with $S \neq P_3$ and let G be a 2-connected, claw-o-heavy graph. Then G being S-c-heavy implies G is Hamiltonian if and only if $S = P_4,\ P_5,\ P_6,\ Z_1,\ Z_2,\ B,\ N$ or W.*

Motivated by Theorems 4 and 5 we naturally propose the notion of c_1-heaviness: we say that a subgraph H of a graph G is c_1-heavy in G if for every maximal clique C of H every non-trivial component of $H - C$ contains a vertex that is super-heavy in G. Graph G is called H-c_1-heavy if every induced subgraph of G isomorphic to H is c_1-heavy.

It is not hard to see that every P_4-c_1-heavy graph is P_5-c_1-heavy and that every P_5-c_1-heavy graph is P_5-o_1-heavy. Furthermore, we notice that every Z_2-c_1-heavy graph is Z_2-o_1-heavy. Thus Theorem 5 implies the following.

Theorem 7. *Let G be a 2-connected, claw-o_1-heavy graph that is not a cycle. If G is S-c_1-heavy, where S is one of $P_4,\ P_5,\ Z_2$, then G is pancyclic.*

The only pair of subgraphs missing in this Theorem and present in Theorem 3 is $\{K_{1,3},\ Z_1\}$. The main result of our paper is the following Theorem, which cannot be deduced from the existing results.

Theorem 8. *Every 2-connected, claw-o_1-heavy and Z_1-c_1-heavy graph that is not a cycle is pancyclic.*

As a consquence of Theorems 3, 7 and 8 we obtain a complete characterization of claw-o_1-heavy and S-c_1-heavy graphs for pancyclicity of 2-connected graphs.

Theorem 9. *Let G be a 2-connected graph which is not a cycle and let S be a connected graph with $S \neq P_3$. Then G being claw-o_1-heavy and S-c_1-heavy implies G is pancyclic if and only if $S = P_4,\ P_5,\ Z_1,\ Z_2$.*

In the next Section we introduce notation used further in the paper and present some of the previous results that will be of use in the proof of Theorem 8. The proof itself is postponed to Section 3.

2. Preliminaries

The subgraph of G induced by the set of vertices $A \subset V(G)$ is denoted $G[A]$. By $G - A$ we denote the subgraph $G[V(G) \setminus A]$. If $A = \{v\}$, we write $G - v$ instead of $G - \{v\}$. Let $A = \{v_1,\ v_2,\ v_3,\ v_4\}$. If $G[A]$ is isomorphic to Z_1, with the set of edges being $\{v_1v_2,\ v_2v_3,\ v_3v_1,\ v_3v_4\}$, we say that $\{v_1,\ v_2;\ v_3,\ v_4\}$ induces a Z_1. Note that if $\{v_1,\ v_2;\ v_3,\ v_4\}$ induces a Z_1 in a

Z_1-c$_1$-heavy graph G, than at least one of the vertices v_1 and v_2 is super-heavy in G.

Let $C = v_1v_2...v_pv_1$ be a cycle. For two positive integers k and m, satisfying $k \leq m \leq p$, by $C[v_k, v_m]$ we denote the set $\{v_k, v_{k+1}, ..., v_m\}$.

Let G be a graph on n vertices. Recall that a vertex $v \in V(G)$ is called heavy, if $d_G(v) \geq n/2$ and super-heavy, if $d_G(v) \geq (n+1)/2$.

Next three lemmas proved to be useful tools in examining heavy graphs with respect to pancyclicity.

Lemma 1 (Benhocine and Wojda [2])**.** *Let G be a graph on $n \geq 4$ vertices and let C be a cycle of length $n-1$ in G. If $d_G(v) \geq n/2$ for $v \in V(G)\setminus V(C)$, then G is pancyclic.*

Lemma 2 (Bondy [3])**.** *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 1$ and $d_G(x) + d_G(y) \geq n+1$, then G is pancyclic.*

Lemma 3 (Ferrara, Jacobson and Harris [9])**.** *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 2$ and $d_G(x) + d_G(y) \geq n+1$, then G is pancyclic.*

The following Lemma gives some information about structure of claw-o_1-heavy graphs (for proof see [7]).

Lemma 4. *Let G be a two-connected, claw-o_1-heavy graph and let $\{r, s\}$ be a vertex-cut in G. Then*

1. *$G - \{r, s\}$ has exactly two components,*
2. *for any distinct neighbours x and x' of r (s) belonging to the same component of $G - \{r, s\}$ either $xx' \in E(G)$ or else $xx' \notin E(G)$ and $d_G(x) + d_G(x') \geq n+1$.*

3. Proof of Theorem 8

The Theorem 8 will be proved by contradiction. Suppose that a graph G on n vertices satisfies the assumptions of the theorem but is not pancyclic. It follows from Theorem 3 that there is either an induced claw or an induced Z_1 in G, implying that there is a super heavy vertex $u \in V(G)$. Consider $G' = G - u$. G' is claw-o-heavy and Z_1-c-heavy. If G' is two-connected, it is Hamiltonian by Theorem 6 and G is pancyclic by Lemma 1, a contradiction. Hence, there is a vertex $v \in V(G)$ such that $\{u, v\}$ is a vertex-cut of G. Lemma 4 implies that $G - \{u, v\}$ consists of exactly two components. Note that G is Hamiltonian by Theorem 6. Let $C = uy_1...y_{h_2}vx_{h_1}...x_1u$ be a

Hamilton cycle in G, where $H_1 = \{x_1, ..., x_{h_1}\}$ and $H_2 = \{y_1, ..., y_{h_2}\}$ are components of $G - \{u, v\}$. Without loss of generality assume $h_1 \leq h_2$.

First we provide some information about H_1.

Claim 1. *There are no super-heavy vertices in H_1.*

Proof. Consider $x \in H_1$, which can be adjacent to at most every other vertex from H_1, u and v. Since $h_1 \leq h_2$, we have $h_1 \leq (n-2)/2$ and so $d_G(x) \leq n/2$. □

Claim 2. *$N_{H_1}[u]$ induces a clique in G.*

Proof. Since the statement is obvious for $h_1 = 1$ and $h_1 = 2$, assume $h_1 \geq 3$. By Claim 1 there are no two vertices in H_1 with sum of degrees greater than n. The Claim follows from Lemma 4. □

The following observation is another simple consequence of Lemma 4.

Claim 3. *Let $y \in N_{H_2}(u)$, $y \neq y_1$. If $y_1 y \notin E(G)$, then y is super-heavy.*

Proof. By Lemma 4 $d_G(y_1) + d_G(y) \geq n + 1$, implying that at least one of these vertices is super-heavy. If y_1 is super-heavy, then $d_G(u)+d_G(y_1) \geq n+1$ and G is pancyclic by Lemma 2, a contradiction. Hence, y is a super-heavy vertex. □

Note that if $d_{H_1}(u) > 1$, then $\{x, x'; u, y_1\}$ induces Z_1 for any two $x, x' \in N_{H_1}(u)$, by Claim 2. Since G is Z_1-c$_1$-heavy, either x or x' must be super-heavy. This contradicts Claim 1. Hence, $d_{H_1}(u) = 1$.

Now consider $y \in N_{H_2}(u) \cap N_{H_2}(y_1)$. Note that y_1 is not super-heavy, since otherwise $d_G(u) + d_G(y_1) \geq n+1$ and G would be pancyclic by Lemma 2. Since $\{y, y_1; u, x_1\}$ induces a Z_1 and G is Z_1-c$_1$-heavy, y is a super-heavy vertex. Together with Claim 3 this implies that every neighbour of u in H_2 other than y_1 is super-heavy. Since u is super-heavy and $d_{H_1}(u) = 1$, we have $d_{H_2}(u) \geq (n+1)/2 - 2$. It follows that there are at least $(n-3)/2$ super-heavy vertices in $C[u, y_{h_2}]$ (with u among them). Since $|C[u, y_{h_2}]| \leq n - 2$, there is a super-heavy pair of vertices in G with distance along the cycle C at most two. Hence, G is pancyclic by Lemma 2 or 3. This final contradiction completes the proof.

□

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