A Fan-type heavy triple of subgraphs for pancyclicity of 2-connected graphs

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Abstract

Graph G of order n is said to be pancyclic if it contains cycles of all lengths k for $k \in \{3, ..., n\}$. A vertex $v \in V(G)$ is called super-heavy if its degree in G is at least $(n+1)/2$. For a given graph S we say that G is S-f_1-heavy if for every induced subgraph K of G isomorphic to S and every two vertices $u, v \in V(K)$, $d_K(u, v) = 2$ implies that at least one of them is super-heavy. For a family of graphs $\mathcal{S}$ we say that G is $\mathcal{S}$-f_1-heavy, if G is S-f_1-heavy for every graph $S \in \mathcal{S}$.

Let H denote the hourglass, a graph consisting of two triangles that have exactly one vertex in common. In this paper we prove that every 2-connected $\{K_{1,3}, P_7, H\}$-f_1-heavy graph on at least nine vertices is pancyclic or missing only one cycle. This result extends the previous work by Faudree, Ryjáček and Schiermeyer.

Keywords: Fan-type heavy subgraph; Hamilton cycle; Pancyclicity

1. Introduction

We use [5] for terminology and notation not defined here. In the paper only finite, simple and undirected graphs are considered.

Let G be a graph on n vertices. G is said to be hamiltonian, if it contains a cycle C_n, and it is called pancyclic, if it contains cycles of all lengths k for $3 \leq k \leq n$. If G does not contain an induced copy of a given graph S, we say that G is S-free. G is called S-f_i-heavy, if for every induced subgraph K of G isomorphic to S and for every two vertices $x, y \in V(K)$ satisfying $d_K(x, y) = 2$, the following inequality holds: $\max\{d_G(x), d_G(y)\} \geq (n+i)/2$.

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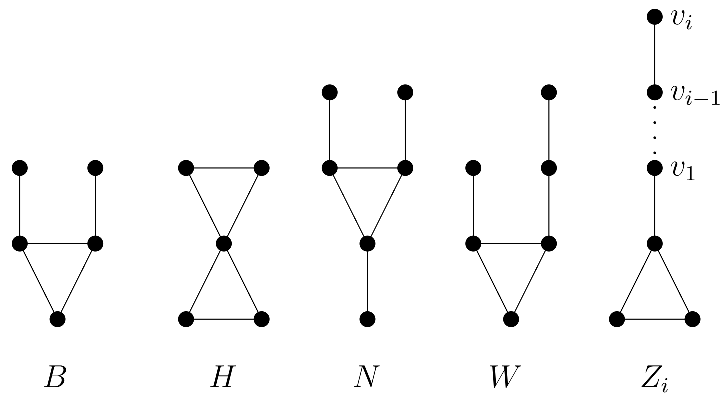

Figure 1: Graphs B (Bull), H (Hourglass), N (Net), W (Wounded) and Z_i

For the sake of simplicity, we write f-heavy instead of f_0-heavy. For a family of graphs $\mathcal{S}$ we say that G is $\mathcal{S}$-free ($\mathcal{S}$-f_i-heavy), if G is S-free (S-f_i-heavy, respectively) for every graph $S \in \mathcal{S}$. The complete bipartite graph $K_{1,3}$ is called a claw.

The notion of f-heaviness was introduced in [12]. It was inspired by the following well-known theorem.

Theorem 1 (Fan [6]). *Let G be a 2-connected graph of order $n \geq 3$. If*

$$d_G(u, v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq n/2$$

for every pair of vertices u and v in G, then G is hamiltonian.

Note that an equivalent formulation of Theorem 1 is that every 2-connected, P_3-f-heavy graph of order $n \geq 3$ is hamiltonian. Clearly, for a given graph S every S-free graph is S-f_i-heavy for every integer i. Hence, it follows from Theorem 1 that every P_3-free graph is hamiltonian (which is not suprising, since the only 2-connected P_3-free graph is a complete graph). Fan's result was extended in 1987 by Wojda and Benhocine (graph F_{4r} appearing in the following theorem consists of a clique on $2r$ vertices that is connected via a perfect matching with r disjoint copies of a path P_2).

Theorem 2 (Benhocine and Wojda [3]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If G is P_3-f-heavy, then G is pancyclic unless $n = 4r$, $r \geq 1$, and G is F_{4r} or else $n \geq 6$ is even and $G = K_{n/2,n/2}$ or $G = K_{n/2,n/2} - e$.*

Since none of the special graphs mentioned in Theorem 2 is P_3-f_1-heavy, it is easy to see that every 2-connected P_3-f_1-heavy graph is pancyclic. It is also not difficult to see that P_3 is the only connected graph S such that every 2-connected S-f_1-heavy graph is pancyclic (for details see [7, Theorem 13]). Bedrossian in his Ph. D. Thesis [1] considered pairs of graphs and managed to characterise all pairs of forbidden subgraphs implying hamiltonicity and pancyclicity of 2-connected graphs. The fact that there are indeed no other pairs of graphs forbidding of which ensures hamiltonicity or pancyclicity was showed a few years later by Faudree and Gould. The graphs B, N, W and Z_i involved in the following theorems are represented on Figure 1.

Theorem 3 (Bedrossian [1]; Faudree and Gould [8])**.** *Let R and S be connected graphs with $R,\ S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R,\ S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4,\ P_5,\ P_6,\ C_3,\ Z_1,\ Z_2,\ B,\ N$ or W.*

Theorem 4 (Bedrossian [1]; Faudree and Gould [8])**.** *Let R and S be connected graphs with $R,\ S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R,\ S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4,\ P_5,\ Z_1$ or Z_2.*

Later both of these results were improved by numerous authors. Note that since every P_3-f-heavy graphs is also S-f-heavy for any connected graph S other than complete graph, Theorem 1 is a corollary from Theorem 5.

Theorem 5 (Ning and Zhang [12])**.** *Let R and S be connected graphs with $R,\ S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R,\ S\}$-f-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4,\ P_5,\ P_6,\ Z_1,\ Z_2,\ B,\ N$ or W.*

Theorem 6. *Let R and S be connected graphs with $R,\ S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R,\ S\}$-f_1-heavy implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and S is one of the following:*
- Z_1 (Bedrossian, Chen and Schelp [2]),
- Z_2, P_4 (Ning [11]),
- P_5 (Wideł [14]).

Triples of forbidden subgraphs with respect to hamiltonian properties have also been extensively examined. One of the many results obtained in this field is the following theorem (see Fig. 1 for the graph H).

Theorem 7 (Faudree et al., Theorem 15 in [7])**.** *Every 2-connected $\{K_{1,3},\ P_7,\ H\}$-free graph on $n \geq 9$ vertices is pancyclic or missing only one cycle.*

Recently, Ning proved the following fact.

Theorem 8 (Ning, [10]). *Every 2-connected $\{K_{1,3},\ P_7,\ H\}$-f-heavy graph is hamiltonian.*

Motivated by Theorems 7 and 8 and by similar results for pairs of forbidden and Fan-type heavy subgraphs, in this paper we prove the following theorem.

Theorem 9. *Let G be a 2-connected, $\{K_{1,3},\ P_7,\ H\}$-f_1-heavy graph. If there is a super-heavy vertex in G, then G is pancyclic.*

Observe that if G is a graph with odd number of vertices, then G being S-f-heavy is equivalent to G being S-f_1-heavy. Hence, Theorem 9 not only extends Theorem 7, but also partialy improves Theorem 8.

In Section 2 we introduce notation used further in the paper and present some of the previous results that will be of use in the proof of Theorem 9. The proof itself is postponed to Section 3.

2. Preliminaries

The neighbourhood of a vertex $v \in V(G)$ is denoted $N_G(v)$. By $N_G[v]$ we denote its closed neighbourhood, that is the set $N_G(v) \cup \{v\}$. If $A \subset V(G)$, then $N_A(v) = N_G(v) \cap A$ and $N_A[v] = N_A(v) \cup \{v\}$.

The subgraph of G induced by the set of vertices $A \subset V(G)$ is denoted $G[A]$. By $G - A$ we denote the subgraph $G[V(G) \setminus A]$. If $A = \{v\}$, we write $G - v$ instead of $G - \{v\}$. Let $A = \{v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6,\ v_7\}$. If $G[A]$ is isomorphic to P_7, where $\{v_1v_2,\ v_2v_3,\ v_3v_4,\ v_4v_5,\ v_5v_6,\ v_6v_7\}$ are the edges of this path, we say that A induces a P_7. If $A = \{v_1,\ v_2,\ v_3,\ v_4\}$ and $G[A]$ is isomorphic to $K_{1,3}$ with v_1v_2, v_1v_3, v_1v_4 being the edges of this claw, we say that $\{v_1;\ v_2,\ v_3,\ v_4\}$ induces $K_{1,3}$ (or induces a claw). Finally, if $A = \{v_1,\ v_2,\ v_3,\ v_4,\ v_5\}$ and $G[A]$ is isomorphic to H, we say that $\{v_1;\ v_2,\ v_3;\ v_4,\ v_5\}$ induces an H, where v_1 is a vertex of degree four in H and the only edges of H that do not contain v_1 are v_2v_3 and v_4v_5 (see Figure 1 for the graph H).

For a cycle $C = v_1v_2...v_pv_1$ we distinguish one of the two possible orientations of C. We write $v_iC^+v_j$ for the path following the orientation of C, i.e., the path $v_iv_{i+1}...v_{j-1}v_j$, and $v_iC^-v_j$ denotes the path from v_i to v_j opposite to the direction of C, that is the path $v_iv_{i-1}...v_{j+1}v_j$. For two positive integers k and m, if there holds $k \leq m \leq p$, then we denote the set $\{v_k,\ v_{k+1},\ ...,v_m\}$ by $C[v_k,\ v_m]$.

Let G be a graph with n vertices. A vertex $v \in V(G)$ is called heavy, if $d_G(v) \geq n/2$ and super-heavy, if $d_G(v) \geq (n+1)/2$. We say that two vertices

u and v form a heavy-pair (super-heavy pair), if both u and v are heavy (super-heavy, respectively).

For two positive integers k and m satisfying $k \leq m$, we say that G contains $[k, m]$-cycles if there are cycles C_k, C_{k+1}, ..., C_m in G.

Lemma 1 (Benhocine and Wojda [3]). *Let G be a graph on $n \geq 4$ vertices and let C be a cycle of length $n-1$ in G. If $d_G(v) \geq n/2$ for $v \in V(G) \backslash V(C)$, then G is pancyclic.*

Lemma 2 (Bondy [4]). *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 1$ and $d_G(x) + d_G(y) \geq n + 1$, then G is pancyclic.*

Lemma 3 (Hakimi and Schmeichel [13]). *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 1$ and $d_G(x)+d_G(y) \geq n$, then G is pancyclic unless G is bipartite or else G is missing only $(n-1)$-cycles.*

Lemma 4 (Ferrara, Jacobson and Harris [9]). *Let G be a graph on n vertices with a Hamilton cycle C. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 2$ and $d_G(x) + d_G(y) \geq n + 1$, then G is pancyclic.*

Lemma 5. *Let G be a graph on n vertices. Let $u, v \in V(G)$ and let i be a positive integer less than $n - 1$. Let X be a set of i vertices $\{x_1, ..., x_i\} \subset V(G)$ that are adjacent neither to u nor to v. Suppose that there are $[n-i+1, n]$ cycles in G and $G' = G - X$ is hamiltonian with a Hamilton cycle C. Then*

1. *if $d_C(u, v) \leq 2$ and $d_G(u) + d_G(v) \geq n - i + 1$, then G is pancyclic,*
2. *if $d_C(u, v) = 1$, $d_G(u) + d_G(v) \geq n - i$ and there is a $(|G'| - 1)$-cycle in G', then G is pancyclic.*

Proof. The first statement is true, since under these assumptions G' is pancyclic by Lemma 2 or 4. If the second case occurs, G' is pancyclic by Lemma 3. Pancyclicity of G' implies pancyclicity of G. □

3. Proof of Theorem 9

Theorem 9 will be proved by contradiction. Suppose that a graph G on n vertices satisfies the assumptions of the theorem but is not pancyclic. Let $u \in V(G)$ be a super-heavy vertex in G. By Theorem 8 G is hamiltonian. It is easy to verify that if $n < 7$, then the edges incident with u and the edges of a hamiltonian cycle of G create cycles of all possible lengths in G. Thus we assume $n \geq 7$. Note that $G-u$ is $\{K_{1,3},\, P_7,\, H\}$-f-heavy. If $G-u$ is also 2-connected, then it is hamiltonian by Theorem 8 and G is pancyclic by Lemma 1, a contradiction. Thus assume that $G-u$ is not 2-connected. This means that there is a vertex $v \in V(G)$ such that $G-\{u,\, v\}$ is not connected. Clearly, since G is hamiltonian, it follows that $G-\{u,\, v\}$ consists of two components. Let $C = uy_1y_2...y_{h_2}vx_{h_1}...x_1u$ be a hamiltonian cycle in G with $H_1 = \{x_1,\, ...,\, x_{h_1}\}$ and $H_2 = \{y_1,\, ...,\, y_{h_2}\}$ being the components of $G-\{u,\, v\}$. Assume, without loss of generality, that $h_1 \leq h_2$. We begin the proof with the following general observations.

Claim 1. *There are no super-heavy vertices in H_1.*

Proof. Consider the vertex $x \in H_1$. Since x can be adjacent to at most u, v and every vertex in $H_1 - x$, it follows that $d_G(x) \leq 2 + h_1 - 1 \leq 2 + (n-2)/2 - 1 = n/2$. Hence, x is not super-heavy. □

Claim 2. $N_{H_2}(u) \subset N_G[y_1]$.

Proof. Suppose that there is a vertex $y \in \{y_3,\, ...,\, y_{h_2}\}$ adjacent to u and not adjacent to y_1. Then $\{u;\, x_1,\, y_1,\, y\}$ induces a claw. Since G is $K_{1,3}$-f_1-heavy and x_1 is not super-heavy, by Claim 1, it follows that y_1 is super-heavy. But then $d_G(u) + d_G(y_1) \geq n+1$. Since $d_C(u,\, y_1) = 1$, G is pancyclic by Lemma 2, a contradiction. □

Claim 3. *There are no super-heavy pairs of vertices with distance one or two along a Hamilton cycle in G.*

Proof. Otherwise G is pancyclic by Lemma 2 or Lemma 4, a contradiction. □

We distinguish two cases, depending on the number of vertices in H_1.

Case 1: $h_1 = 1$.

We begin the proof of this case with establishing some facts regarding the neighbourhood of u in G. Then we use them to show that there exists an induced P_7 in G with most of its vertices lying close to each other on C. This leads to a contradiction with Claim 3.

Claim 4. $uv \in E(G)$.

Proof. Suppose that the claim is not true. Since u is super-heavy and x_1 is a neighbour of u that is not adjacent to y_1, Claim 2 implies that $d_G(y_1) \geq (n-1)/2$. Hence, $d_G(u) + d_G(y_1) \geq n$.

Suppose that u is adjacent to y_2. Since G is hamiltonian and uy_2C^+u is an $(n-1)$-cycle in G, this implies that G is neither bipartite nor missing $(n-1)$-cycle, and so G is pancyclic by Lemma 3, a contradiction. Hence, $uy_2 \notin E(G)$. Since y_2 is adjacent to y_1, Claim 2 now implies that $d_G(y_1) \geq (n+1)/2$. Thus $\{u, y_1\}$ is a super-heavy pair of vertices, in contradiction with Claim 3. □

Claim 5. $N_{H_2}[u] = N_G[y_1]$.

Proof. Suppose that the claim is not true. Then, by Claim 2 and Claim 4, either there is a vertex $y \in H_2$ adjacent to y_1 and not adjacent to u or else $vy_1 \in E(G)$. In either case it follows that $d_G(y_1) \geq (n+1)/2 - 1$ and so $d_G(u) + d_G(y_1) \geq n$. Since G is hamiltonian and uC^+vu is a cycle of length $n-1$, G is neither bipartite nor missing $(n-1)$-cycles. Lemma 3 implies that G is pancyclic, a contradiction. □

Claim 6. $N_G[v] = N_G[u] \setminus \{y_1\}$.

Proof. First we show the inclusion $N_G[u] \setminus \{y_1\} \subset N_G[v]$. Consider $y \in N_{H_2}(u) \setminus \{y_1\}$. If y is not adjacent to v, then, by Claim 5, Claim 4 and the assumptions of this case, the vertices from the set $\{u; v, x_1; y_1, y\}$ induce an H. Since G is H-f_1-heavy, either x_1 or y_1 is super-heavy. But u is a super-heavy vertex with $d_C(u, x_1) = 1$ and $d_C(u, y_1) = 1$, a contradiction with Claim 3.

Hence, every neighbour of u in H_2 other than y_1 is also a neighbour of v. Since u is super-heavy and both u and v are adjacent to x_1, this implies that $d_G(v) \geq (n+1)/2 - 1$. Suppose that there is a vertex in $N_G(v)$ that is not adjacent to u. Then $d_G(v) \geq (n+1)/2$ and $\{u, v\}$ is a super-heavy pair with $d_C(u, v) = 2$, contradicting Claim 3. The claim follows. □

Claim 7. $d_G(v) \geq (n+1)/2 - 1$.

Proof. The statement follows immediately from Claim 6 and the fact that u is super-heavy. □

Claim 8. *There are* $[n-3, n]$*-cycles in* G.

Proof. Obviously, C is a hamiltonian cycle in G. By Claims 4, 5 and 6 $uy_2C^+y_{h_2}u$ is a cycle of length $n-3$, uy_2C^+vu is an $(n-2)$-cycle and uy_2C^+u is a cycle of length $n-1$. □

Claim 9. *Let* $A = \{y_{a+1}, ..., y_{a+p}\}$ *be a maximal (inclusion-wise) set of consecutive non-neighbours of* u *in* H_2. *Then* $p \geq 4$. *Furthermore, if* $y_by_c \in E(G)$ *for some* $b \in \{a, a+1, ..., a+p-1\}$ *and* $c \in \{b+1, ..., a+p\}$, *then either* $c = b+1$ *or else* $c \geq b+5$.

Proof. Suppose that the claim is not true. Note that every non-neighbour of u is not adjacent to v by Claim 6. Recall that $d_G(u) + d_G(v) \geq n$, by Claim 7, and that there are $[n-3, n]$-cycles in G, by Claim 8. Suppose that $p < 4$ and consider $G' = G - A$. Note that the cycle $C' = y_{a+p+1}C^+uy_aC^-y_1y_{a+p+1}$ is a hamiltonian cycle in G'. Since $d_{C'}(u, v) = 2$, pancyclicity of G follows from Lemma 5 (for $X = A$ and $i = p \leq 3$).

Now suppose that there is an edge y_by_{b+k} for some $y_b \in A \cup \{y_a\}$ and $y_{b+k} \in A$, with $k \in \{2, 3, 4\}$. Applying Lemma 5 for u, v and $X = \{y_{b+1}, ..., y_{b+k-1}\}$ yields pancyclicity of G, since in $G' = G - X$ there is a hamiltonian cycle $uC^+y_by_{b+k}C^+u$. A contradiction. □

Note that if u is adjacent to all vertices in G, then G is clearly pancyclic, a contradiction. Hence, there is a vertex $y_i \in H_2$ adjacent to u such that $y_{i+1} \in H_2$ and $uy_{i+1} \notin E(G)$. It follows from Claim 9 that the set $\{x_1, u, y_i, y_{i+1}, y_{i+2}, y_{i+3}, y_{i+4}\}$ induces P_7. Since G is P_7-f_1-heavy and x_1 is not super-heavy by Claim 1, the vertex y_i is super-heavy. But then, by Claim 3, neither y_{i+1} nor y_{i+2} can be super-heavy, implying that $\{y_{i+3}, y_{i+4}\}$ is a super-heavy pair. Since $d_C(y_{i+3}, y_{i+4}) = 1$, this contradicts Claim 3 and completes the proof of this case.

Case 2: $h_1 \geq 2$.

Similarly to the proof of the previous case, firstly the neighbourhood of the vertex u is examined. Then we focus on the vertex v and arrive at the final contradiction.

Claim 10. *No two neighbours of u in H_1 are adjacent.*

Proof. Suppose that the claim is not true, i.e., suppose that there are two vertices $x_a, x_b \in N_{H_1}(u)$ such that x_ax_b is an edge in G. If u is adjacent to some vertex $y \in H_2$ other than y_1, then, by Claim 2, the set $\{u; x_a, x_b; y_1, y\}$ induces an H. Since G is H-f_1-heavy and neither x_a nor x_b is super-heavy (by Claim 1), it follows that y_1 is super-heavy. But then $\{u, y_1\}$ is a super-heavy pair of vertices, in contradiction to Claim 3. Thus $N_{H_2}(u) = \{y_1\}$.

Since u is super-heavy and can be adjacent to at most y_1, v and all vertices of H_1, it follows that $(n+1)/2 \leq d_G(u) \leq h_1 + 2$. This, together with the fact that $n = h_1 + h_2 + 2$ and $h_1 \leq h_2$, implies $(n-3)/2 \leq h_1 \leq (n-2)/2$. Whether n is even and equal to $2k$ or odd, and equal to $2k+1$, this implies that $h_1 = k - 1$. Now in order for u to be super-heavy, its neighbourhood must be $N_G(u) = H_1 \cup \{y_1, v\}$, implying the existence of $[3, h_1 + 2]$-cycles in G, which can be rewritten as $[3, k+1]$ cycles. Note that $C' = uC^+vu$ is a cycle of length $n - h_1 = n - k + 1 \leq k + 2$. By appending the neighbours of

u from H_2 along the orientation of the cycle C to C', one-by-one, we obtain $[k+2, n]$-cycles. Hence G is pancyclic, a contradiction. □

Claim 11. $N_{H_1}(u) = \{x_1\}$.

Proof. Suppose that this is not the case. Then u is adjacent to some vertex $x_k \in H_1$ other than x_1. Claim 10 implies that $\{u;\, x_1,\, x_k,\, y_1\}$ induces a claw. Since neither x_1 nor x_k is super-heavy, by Claim 1, this contradicts G being $K_{1,3}$-f_1-heavy. □

Claim 12. $uv \in E(G)$.

Proof. Suppose that the claim is not true, i.e., that u is not adjacent to v. Since $N_{H_1}(u) = \{x_1\}$ (by Claim 11), it follows that $d_{H_2}(u) \geq (n+1)/2 - 1$ and, by Claim 2, $d_G(y_1) \geq (n+1)/2 - 1$. Suppose that $uy_2 \notin E(G)$. Since y_2 is adjacent to y_1, now Claim 2 implies that $d_G(y_1) \geq (n+1)/2$ and so $\{u,\, y_1\}$ is a super-heavy pair with $d_C(u,\, y_1) = 1$. This contradicts Claim 3. Thus $uy_2 \in E(G)$. Note that $d_G(u) + d_G(y_1) \geq n$. Since G is hamiltonian and uy_2C^+u is an $(n-1)$-cycle in G, G is pancyclic by Lemma 3. □

Claim 13. *v is not adjacent to y_1. Furthermore, $N_G[u] \setminus \{y_1\} \subset N_G[v]$.*

Proof. It will be first shown that vy_1 is not an edge in G. To do this, suppose the contrary. Since $d_{H_2}(y_1) \geq (n+1)/2 - 3$, by Claim 2, Claim 11 and Claim 12, and both uy_1 and vy_1 are edges in G, it follows that $d_G(y_1) \geq (n-1)/2$. If $uy_2 \notin E(G)$, then $\{u,\, y_1\}$ is a super-heavy pair of vertices, in contradiction with Claim 3. Thus $uy_2 \in E(G)$. But now uy_2C^+u is a cycle of length $n-1$. This implies, together with the existence of a triangle uy_1y_2u in G and the fact that $d_G(u) + d_G(y_1) \geq n$ that G is pancyclic, by Lemma 3. A contradiction. Thus $vy_1 \notin E(G)$. It follows that $vx_1 \in E(G)$, since otherwise $\{u;\, x_1,\, v,\, y_1\}$ would induce a claw with neither x_1 nor y_1 being super-heavy (by Claim 3 and the fact that u is super-heavy). Thus $N_{H_1}[u] \subset N_G[v]$, by Claim 11 and Claim 12.

Consider now a vertex $y \in N_{H_2}(u)$ other than y_1. If y is not adjacent to v, then $\{u;\, v,\, x_1;\, y_1,\, y\}$ induces an H, by Claim 2. Since u is super-heavy, it follows from Claim 3 that neither x_1 nor y_1 is super-heavy. This contradicts G being H-f_1-heavy. Hence, $N_{H_2}(u) \setminus \{y_1\} \subset N_G[v]$. The claim follows. □

Claim 14. *u is adjacent to y_2.*

Proof. Suppose that the claim is not true. Since $d_{H_2}(u) \geq (n+1)/2 - 2$, by Claim 11 and Claim 12, it follows from Claim 2 and the fact that y_2 is a neighbour of y_1 but not a neighbour of u that $d_G(y_1) \geq (n+1)/2 - 1$. Thus

$d_G(u) + d_G(y_1) \geq n$ and so, by Lemma 3, G is either pancyclic, bipartite or else missing only $(n-1)$-cycle. Suppose that there are no cycles of length $n-1$ in G. Then $h_1 \geq 3$, since otherwise, by Claim 13 and the assumptions of this case, the cycle vx_1C^+v would be such a cycle. Furthermore, none of the edges x_1x_3, y_1y_3 and uy_2 can exist. This implies, by Claim 2, that $uy_3 \notin E(G)$ and so $\{x_3, x_2, x_1, u, y_1, y_2, y_3\}$ induces a P_7. Since there are no super-heavy vertices in H_1, by Claim 1, this contradicts G being P_7-f_1-heavy. Hence, there exists an $(n-1)$-cycle in G. Since G contains $(n-1)$-cycle and n-cycle, it is not bipartite, and so it is pancyclic. A contradiction. □

Claim 15. *v is a super-heavy vertex.*

Proof. Clearly, the vertex u has at least $(n+1)/2-2$ neighbours in H_2, by Claims 11 and 12. Claim 13 implies that $d_{H_2}(v) \geq (n+1)/2-3$. Since v is also adjacent to x_{h_1}, u (by Claim 12) and x_1 (by Claim 13), the claim follows. □

Claim 16. *$N_{H_2}(v) \subset N_G[y_{h_2}]$.*

Proof. Suppose that the claim is not true. Then there is a vertex $y \in H_2 - y_{h_2}$ that is adjacent to v and not adjacent to y_{h_2}. It follows that the set $\{v;\, x_{h_1}, y_{h_2}, y\}$ induces a claw in G. Since x_{h_1} is not super-heavy, by Claim 1, and G is $K_{1,3}$-f$_1$-heavy, this implies that y_{h_2} is super-heavy. But now $\{v, y_{h_2}\}$ is a super-heavy pair of vertices, in contradiction with Claim 3. □

Recall that at the beginning of the proof we assumed that $n \geq 7$. This implies in particular that $h_2 \geq 3$. Now it follows from Claims 13, 14 and 16 that vx_1, vy_2 and $y_2y_{h_2}$ are edges in G. Hence, depending on the existence of the edge $x_1x_{h_1}$, either the set $\{v;\, x_1, x_{h_1}, y_{h_2}\}$ induces a claw or else the set $\{v;\, x_1, x_{h_1};\, y_2, y_{h_2}\}$ induces an H. Since y_{h_2} cannot be super-heavy, by Claims 3 and 15, and neither x_1 nor x_{h_1} is super-heavy, by Claim 1, this contradicts G being $\{K_{1,3}, H\}$-f$_1$-heavy. This final contradiction completes the proof.

□

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