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Heavy subgraphs and pancyclicity

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Contents

1 Introduction

The main aim of the thesis is to give new sufficient conditions for existence of cycles in 2-connected simple graphs. The conditions under consideration involve imposing some requirements on degrees of some of the graphs vertices in order to entail its hamiltonicity or the existence of cycles of all possible lengths. The results obtained extend some classical degree-type conditions for hamiltonicity and pancyclicity, as well as some conditions expressed in terms of forbidden subgraphs.

All of the notions and symbols not defined explicitly in the thesis are used according to [8]. For a graph G we denote its set of vertices and set of edges by $V(G)$ and $E(G)$, respectively. The neighbourhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$ and the number $d_G(v)$ of its elements is called the degree of v. The minimum degree of the vertices of G is denoted $\delta(G)$. If there are cycles of all possible lengths in G (i.e., cycles of lengths 3, 4, ..., $|V(G)|$), then G is said to be pancyclic. The distance $d_G(u, v)$ between two vertices u and v of a connected graph G is the length of the shortest path connecting them (i.e., the number of the edges of such a path). By P_n we denote a path of order n. Graph obtained from G by removing one of its edges is denoted by $G - e$.

A cycle passing through all of the graph's vertices is called its hamiltonian cycle (or Hamilton cycle). This specific cycle owes its name to sir William Rowan Hamilton who, in a letter to a friend from 1856, described a game played on a regular dodecahedron. The aim of the game was to create a path beginning and ending in a given vertex that passes through all of the other vertices, while visiting each of them exactly once (in 1859 Hamilton was able to sell the game to a London game dealer for 25 pounds; for a more complete description of the game and of its mathematical model see [1], p. 262). Since the problem of determining whether or not there is a hamiltonian cycle in a given graph is NP-complete, the knowledge of conditions ensuring hamiltonicity, satisfiability of which can be easily verified is desirable. Some of the most recent results in this field can be found in surveys [28] and [38]. One of the first results connecting the graph's vertices' degrees with the existence of a Hamilton cycle is the following theorem by Dirac from 1952.

Theorem 1.1 (Dirac [15]). *Let G be a graph of order $n \geq 3$. If the minimal degree of G satisfies $\delta(G) \geq n/2$, then G is hamiltonian.*

Eight years later Ore showed that the Dirac's condition can be weakened.

Theorem 1.2 (Ore [43]). *Let G be a graph of order n. If for every pair of its non-adjacent vertices the sum of their degrees is not less than n, then G is hamiltonian.*

In 1984 Fan gave an even more general result for 2-connected graphs. Note that the assumption of a graph being 2-connected is not at all limiting, since 2-connectedness is a necessary condition for hamiltonicity.

Theorem 1.3 (Fan [17]). *Let G be a 2-connected graph of order $n \geq 3$. If*

$$d_G(u, v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq n/2$$

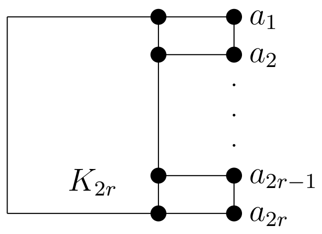

Fig. 2.1: The Fan's graph F_{4r}.

for every pair of vertices u and v in G, then G is hamiltonian.

Bondy noticed that from the existing sufficient conditions for hamiltonicity one can deduce even more information regarding graph's cycle structure. In [7] he posed the so-called Bondy's meta-conjecture which states that almost every non-trivial sufficient condition for hamiltonicity ensures in fact pancyclicity, possibly besides a finite number of exceptional graphs. The following results, first of which extends Theorem 1.2 and the other one extending Theorem 1.3, support this meta-conjecture (graph F_{4r} appearing in the following consists of a clique on $2r$ vertices that is connected via a perfect matching with r disjoint copies of a path P_2; it is presented on Figure 2.1).

Theorem 1.4 (Bondy [6])**.** *Let G be a graph of order $n \geq 3$. If for every pair of its non-adjacent vertices the sum of their degrees is not less than n, then G is pancyclic unless n is even and $G = K_{n/2,\,n/2}$.*

Theorem 1.5 (Benhocine and Wojda [4])**.** *Let G be a 2-connected graph of order $n \geq 3$. If*

$$d_G(u,\,v) = 2 \Rightarrow \max\{d_G(u),\, d_G(v)\} \geq n/2$$

for every pair of vertices u and v in G, then G is pancyclic unless $n = 4r$, $r \geq 1$, and G is F_{4r}, or else $n \geq 6$ is even and $G = K_{n/2,\,n/2}$ or $G = K_{n/2,\,n/2} - e$.

It is easy to see that a slight strengthening of the assumptions of the above theorems results in sufficient conditions for pancyclicity that are free of exceptions.

Corollary 1.1. *Let G be a graph of order $n \geq 3$. If for every pair of its non-adjacent vertices the sum of their degrees is not less than $n+1$, then G is pancyclic.*

Corollary 1.2. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$d_G(u,\,v) = 2 \Rightarrow \max\{d_G(u),\, d_G(v)\} \geq (n+1)/2$$

for every pair of vertices u and v in G, then G is pancyclic.

The above corollaries and Theorem 1.5 constitute the first of the basic motivations for our research. Theorems due to the author of the thesis are indicated with initials WW and presented with full proofs.

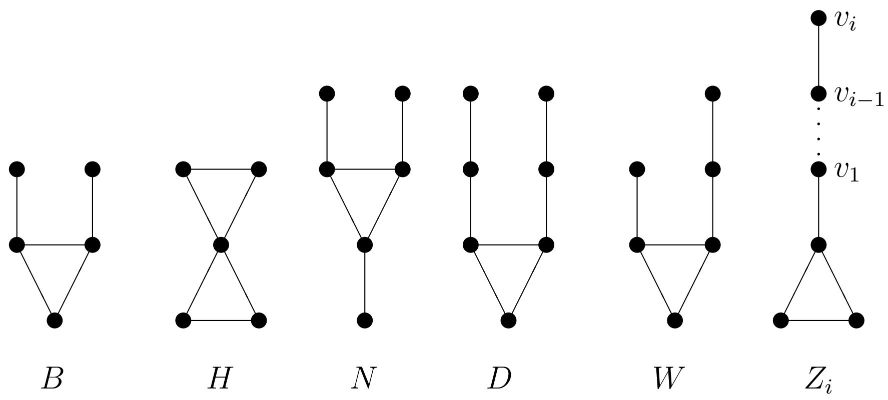

Fig. 2.2: Graphs B (bull), H (hourglass), N (net), D (deer), W (wounded) and Z_i.

1.1 Forbidden subgraphs

The second of our motivations were the results connecting the properties of hamiltonicity and pancyclicity of 2-connected graphs with their subgraphs. A subgraph of G induced by a set of vertices $A \subset V(G)$ is a subgraph of G whose set of vertices is A and whose set of edges consists of all the edges of G whose both endvertices belong to A. If there are no induced copies of a graph S in G, then G is said to be S-free. If one demands G being S-free (or being $\mathcal{S}$-free for a family of graphs $\mathcal{S}$), then S is said to be forbidden in G (respectively, the family $\mathcal{S}$ is forbidden in G). The complete bipartite graph $K_{1,3}$ is called a claw. All of the special graphs that appear in the results presented further in the thesis are represented on Figure 2.2.

It is easy to see, that every 2-connected P_3-free graph is a complete graph and as such it is both hamiltonian and pancyclic. A fact that is a bit harder to prove is that the path P_3 is the only graph forbidding of which in 2-connected graph ensures its hamiltonicity, and the only one forbidding of which implies pancyclicity (for the proof see [20]). The next natural step in examining connections between induced subgraphs and the existence of cycles in graphs was to consider pairs of forbidden subgraphs, with P_3 excluded. The first result of this type was published in 1974 and is due to Goodman and Hedetniemi.

Theorem 1.6 (Goodman, Hedetniemi [27]). *Every 2-connected $\{K_{1,3}, Z_1\}$-free graph is hamiltonian.*

Note that every C_3-free graph is also Z_1-free. Hence, it follows from the above theorem that every 2-connected $\{K_{1,3}, C_3\}$-free graph is hamiltonian. In fact, one can easily check that the only graphs satisfying this condition are cycles of order at least four. Theorem 1.6 was improved a few years later in the following ways.

Theorem 1.7 (Duffus, Gould, Jacobson [16]). *Every 2-connected $\{K_{1,3}, N\}$-free graph is hamiltonian.*

Theorem 1.8 (Gould, Jacobson [29]). *Every 2-connected $\{K_{1,3}, Z_2\}$-free graph is either pancyclic or a cycle.*

The next pair of forbidden subgraphs ensuring hamiltonicity of 2-connected graphs was presented in 1990 by Broersma and Veldman.

Theorem 1.9 (Broersma, Veldman [9]). *Every 2-connected $\{K_{1,3}, P_6\}$-free graph is hamiltonian.*

In his Ph. D. thesis from 1991, Bedrossian gathered the above results and presented also the last pair of forbidden subgraphs for hamiltonicity of 2-connected graphs. The fact that forbidding any other pair of subgraphs indeed does not imply hamiltonicity was showed six years later by Faudree and Gould. These results can be presented in the following form.

Theorem 1.10 (Bedrossian [2]; Faudree, Gould [20]). *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3, P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.*

It was also showed by Bedrossian in [2] that forbidding the pair $\{K_{1,3}, P_5\}$ in a 2-connected graph G implies, similarly to Theorem 1.8, that G is either pancyclic or else a short cycle. Faudree and Gould proved that these two pairs of subgraphs are the only ones forbidding of which in 2-connected graphs (other than cycles) implies pancyclicity. Since the path P_4 is an induced subgraph of P_5 and Z_1 is an induced subgraph of Z_2, we state this fact as follows.

Theorem 1.11 (Bedrossian [2]; Faudree, Gould [20]). *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph which is not a cycle. Then G being $\{R, S\}$-free implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

Theorems 1.10 and 1.11 provide a complete characterization of forbidden pairs of subgraphs for hamiltonicity and pancyclicity of 2-connected graphs. List of all forbidden triples ensuring hamiltonicity which are of the form $\{K_{1,3}, R, S\}$ can be found in [10], and of all the triples that do not contain a claw in [21]. Two of these triples are of interest for us (for graphs H and D see Figure 2.2).

Theorem 1.12 (Faudree et al. [19]; Brousek [10]). *Every 2-connected, $\{K_{1,3}, P_7, H\}$-free graph is hamiltonian.*

Theorem 1.13 (Broersma, Veldman [9]; Brousek [10]). *Every 2-connected, $\{K_{1,3}, P_7, D\}$-free graph is hamiltonian.*

These two particular triples were examined a few years before the publication of Brousek's result by Faudree, Ryjáček and Schiermeyer. They showed in [19] that in graphs of order big enough forbidding of these triples ensures in fact pancyclicity, perhaps with cycles of exactly one length missing.

Theorem 1.14 (Faudree et al., Theorem 15 in [19]). *Every 2-connected, $\{K_{1,3}, P_7, H\}$-free graph on $n \geq 9$ vertices is pancyclic or missing only one cycle.*

Theorem 1.15 (Faudree et al., Corollary F in [19])**.** *Every* 2*-connected,* $\{K_{1,3}, P_7, D\}$*-free graph on* $n \geq 14$ *vertices is pancyclic*

In the results presented so far the sufficient conditions for hamiltonicity and pancyclicity, both in terms of degrees and in terms of forbidden subgraphs, were quite strong. In order to weaken the conditions of the first type, one can try to limit the number of vertices on which a high degree requirement is imposed. Weakening of the forbidden subgraph-type conditions can be achieved by allowing the forbidden subgraphs to be present in a graph, but with some degree conditions imposed on their vertices. Theorems 1.2 and 1.3 are natural inspiration for a suitable choice of such conditions.

We finish this subsection with a short digression. The thesis is exclusively devoted to 2-connected graphs, because these graphs were the object of our research. Interested reader can find in [30] a complete characterization of forbidden pairs of subgraphs for pancyclicity of 3-connected graphs. Some partial results concerning forbidden subgraph-type conditions for hamiltonicity in 3-connected graphs can be found in [39], [33] [25] or [53]. For similar results regarding 4-connected graphs see [44], [26] (for hamiltonicity) or [26], [24] and [22] (for sufficient conditions for pancyclicity). Since the aim of this thesis is by no means to present the state of the art in the field of forbidden subgraph-type conditions for the existence of cycles in graphs, we do not present the main results of the above mentioned articles.

1.2 Fan-type heavy subgraphs

In his paper from 1984 Fan actually proved a result more general than Theorem 1.3.

Theorem 1.16 (Fan [17])**.** *Let* G *be a* 2*-connected graph with* n *vertices and let* $3 \leq k \leq n$*. If*

$$d_G(u, v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq k/2$$

for every pair of vertices u *and* v *in* G*, then there is a cycle of length at least* k *in* G*.*

Imposing the above degree condition on subgraphs appearing in Theorems 1.10 and 1.11 is one of the possible ways of generalizing these theorems. This idea was explored by many researchers, using various terminology and notations. Before we state their results, we introduce a notion that encapsulates these different notations.

Definition 1. Let $\mathcal{S}$ be a family of graphs and let k be a positive integer. We say that a graph G *satisfies Fan's condition with respect to* $\mathcal{S}$ *with constant* k, if for every induced subgraph S of G isomorphic to any of the graphs from $\mathcal{S}$ the following holds:

$$\forall\, u, v \in V(S)\colon d_S(u, v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq k/2.$$

By $\mathcal{F}(\mathcal{S}, k)$ we denote the family of graphs satisfying the Fan's condition with respect to $\mathcal{S}$ with constant k. If $\mathcal{S}$ consists of one element, say S, we write $\mathcal{F}(S, k)$ instead of $\mathcal{F}(\{S\}, k)$. Note that given a family of graphs $\mathcal{S}$ and a constant k, every $\mathcal{S}$-free graph satisfies Fan's condition with respect to $\mathcal{S}$ with constant k. It is also clear that if $G \in \mathcal{F}(P_3, k)$, then

$G \in \mathcal{F}(\mathcal{S}, k)$ for any connected graph S. The authors of [3] were first to impose the Fan's condition on one of the pairs of subgraphs that appear in Theorems 1.10 and 1.11. They obtained the following results.

Theorem 1.17 (Bedrossian, Chen and Schelp [3]). *Let G be a 2-connected graph with n vertices. If $3 \leq k \leq n$ and $G \in \mathcal{F}(\{K_{1,3}, Z_1\}, k)$, then there is a cycle of length at least k in G.*

Theorem 1.18 (Bedrossian, Chen and Schelp [3]). *Let G be a 2-connected graph of order $n \geq 3$ which is not a cycle. If $G \in \mathcal{F}(\{K_{1,3}, Z_1\}, n)$, then G is pancyclic unless $n = 4r$, $r \geq 1$, and G is F_{4r}, or else $n \geq 6$ is even and $G = K_{n/2, n/2}$ or $G = K_{n/2, n/2} - e$.*

A natural next step towards extending Theorems 1.10 and 1.11 (as well as Theorems 1.5 and 1.16) in the direction indicated by Theorems 1.17 and 1.18 was to impose Fan's condition on the pair $\{K_{1,3}, P_4\}$.

Theorem 1.19 (WW [51]). *Let G be a 2-connected graph of order n. If $3 \leq k \leq n$ and $G \in \mathcal{F}(\{K_{1,3}, P_4\}, k)$, then there is a cycle of length at least k in G.*

Theorem 1.20 (WW [51]). *Let G be a 2-connected graph of order $n \geq 3$. If $G \in \mathcal{F}(\{K_{1,3}, P_4\}, n)$, then G is pancyclic unless $n = 4r$, $r \geq 1$, and G is F_{4r}, or else $n \geq 6$ is even and $G = K_{n/2, n/2}$ or $G = K_{n/2, n/2} - e$.*

In Chapter 3 the proof of Theorem 1.19 is presented. The proof of Theorem 1.20 can be found in Chapter 4. Clearly, Theorem 1.16 is a corollary from Theorem 1.19 and Theorem 1.5 follows from Theorem 1.20.

Most of the papers devoted to the problem of improving Bedrossian's results involve the Fan's condition with a constant k being equal to the order of the graph. Note that the pair $\{K_{1,3}, C_3\}$ which appears in Theorem 1.10 is missing in the following result. This is due to the fact that for every integer $m \geq 2$ every graph satisfies Fan's condition with respect to the complete graph K_m with any real number k.

Theorem 1.21. *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph of order n. Then $G \in \mathcal{F}(\{R, S\}, n)$ implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and S is one of the following:*
- P_4, P_5, P_6 (Chen, Wei and X. Zhang [14]),
- Z_1 (Bedrossian, Chen and Schelp [3]),
- B (G. Li, Wei and Gao [37]),
- N (Chen, Wei and X. Zhang [13]),
- Z_2, W (Ning and S. Zhang [42]).

In light of Theorems 1.5, 1.18 and 1.19 it is clear that in general case imposing the Fan's condition on the pairs of subgraphs from Theorem 1.11 with a constant equal to the order of the graph is not enough for ensuring pancyclicity. The existence of cycles of all possible lengths is entailed by imposing a slightly stronger condition.

Theorem 1.22. *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph of order n which is not a cycle. Then $G \in \mathcal{F}(\{R, S\}, n+1)$ implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and S is one of the following:*
- Z_1 (Bedrossian, Chen and Schelp [3]),
- Z_2, P_4 (Ning [41]),
- P_5 (WW [47]).

The proof of the last part of the above theorem, that is the fact that every 2-connected graph belonging to $\mathcal{F}(\{K_{1,3}, P_5\}, n+1)$ other than a cycle is pancyclic is not included in the thesis itself, since it has been already published and the general idea of the proof is also exploited in Chapter 5. However, for the convenience of interested readers we attach a copy of the paper containing the proof. Note also, that from the exceptional non-pancyclic graphs mentioned in Theorem 1.20 only the cycle $K_{2,2}$ satisfies Fan's condition with respect to $\{K_{1,3}, P_4\}$ with constant $n+1$. Hence, the part of Theorem 1.22 regarding the pair $\{K_{1,3}, P_4\}$ can be deduced from Theorem 1.20.

The results presented so far suggest posing the following conjectures.

Conjecture 1.1. *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph with n vertices. If $3 \leq k \leq n$, then $G \in \mathcal{F}(\{R, S\}, k)$ implies that there is a cycle of length at least k in G if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3, P_4, P_5, P_6, Z_1, Z_2, B, N$ or W.*

Conjecture 1.2. *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph of order n other than C_n, $F_{4(n/4)}$, $K_{n/2,n/2}$ and $K_{n/2,n/2} - e$. Then $G \in \mathcal{F}(\{R, S\}, n)$ implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

Imposing an appropriate Fan's condition on some triples of subgraphs also yielded new sufficient conditions for hamiltonicity of 2-connected graphs. The following results extend Theorems 1.12 and 1.13.

Theorem 1.23 (Ning [40]). *Let G be a 2-connected graph of order $n \geq 3$. If $G \in \mathcal{F}(\{K_{1,3}, P_7, H\}, n)$, then G is hamiltonian.*

Theorem 1.24 (Ning [40]). *Let G be a 2-connected graph of order $n \geq 3$. If $G \in \mathcal{F}(\{K_{1,3}, P_7, D\}, n)$, then G is hamiltonian.*

Motivated by Theorems 1.23 and 1.24 and by similar results for pairs of forbidden and Fan-type heavy subgraphs, we extended Theorems 1.14 and 1.15 in the following way.

Theorem 1.25 (WW [49]). *Let G be a 2-connected graph of order $n \geq 3$. If $G \in \mathcal{F}(\{K_{1,3}, P_7, H\}, n+1)$ and there is a vertex of degree at least $(n+1)/2$ in G, then G is pancyclic.*

Theorem 1.26 (WW [52]). *Let G be a 2-connected graph of order $n \geq 14$. If $G \in \mathcal{F}(\{K_{1,3}, P_7, D\}, n+1)$, then G is pancyclic.*

As the proofs of the both above theorems share the same general framework, instead of presenting them separately, in Chapter 5 we give a proof of the following theorem.

Theorem 1.27 (WW)**.** *Let G be a 2-connected graph with n vertices. If $G \in \mathcal{F}(\{K_{1,3}, P_7\}, n+1)$ and*

1. *$n \geq 14$ and $G \in \mathcal{F}(D, n+1)$, or*

2. *$G \in \mathcal{F}(H, n+1)$ and there is a vertex of degree at least $(n+1)/2$ in G,*

then G is pancyclic.

Before we present another type of heavy subgraphs, the one inspired by Theorem 1.2, we note that the Fan-type degree conditions can be relaxed further. Instead of demanding from some of the vertices of a graph of order n to have degree not less than $n/2$ or $(n+1)/2$, one can require that their implicit degrees (introduced in [54]) satisfy these inequalities. Since the implicit degree of a vertex is not less than its degree, this is a weaker requirement. Using these idea the authors of [12], [11] and [50] improved Theorems 1.23 and 1.24. Since our results presented in [50] are sufficient conditions for hamiltonicity, and the main focus of the thesis are sufficient conditions for pancyclicity, they are not included in the thesis.

1.3 Ore-type heavy subgraphs

Another possible approach to weakening of the assumptions of Theorems 1.10 and 1.11 is to impose on the subgraphs they involve an Ore-type degree condition. A specific type of Ore-type heavy subgraphs was first introduced in [46]. The authors of [36] extended this idea in the following way.

Definition 2. Graph G is said to be *S-o-heavy* (*S-o_1-heavy*) if in every induced subgraph of G isomorphic to S there are two non-adjacent vertices with the sum of their degrees in G at least $|V(G)|$ ($|V(G)|+1$).

Clearly, every S-free graph is trivially S-*o*-heavy. Hence the following theorem extends Bedrossian's result.

Theorem 1.28 (B. Li, Ryjáček, Wang, S. Zhang [36])**.** *Let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-o-heavy implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = C_3, P_4, P_5, Z_1, Z_2, B, N$ or W.*

Note that the only pair of subgraphs that appears in Theorem 1.10 and does not appear here is $\{K_{1,3}, P_6\}$. The authors of the above Theorem present in [36] an example of a non-hamiltonian $\{K_{1,3}, P_6\}$-*o*-heavy (and even claw-free, P_6-*o*-heavy) graph. It is denoted as G_1 on Figure 2.3. For V_1, V_2 and V_3 being a balanced partition of its clique K_{3p} (with $p \geq 5$) each of the vertices x_i, for $i \in \{1, 2, 3\}$, is joined via an edge with all of the vertices from the sets V_j for $j \neq i$.

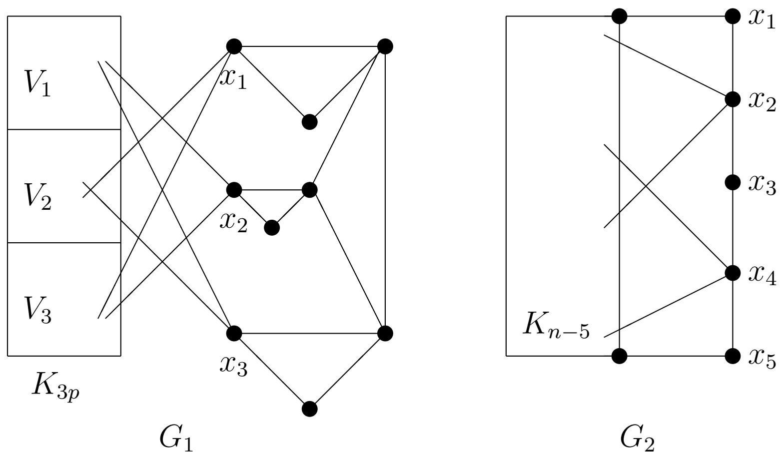

Fig. 2.3

Similarly to the case of Fan-type heavy subgraphs, imposing a slightly stronger version of Ore-type heaviness on the pairs of forbidden subgraphs from Theorem 1.11 yields a sufficient condition for pancyclicity.

Theorem 1.29 (B. Li, Ning, Broersma, S. Zhang [35]). *Let G be a 2-connected graph which is not a cycle and let R and S be connected graphs with $R \neq P_3$, $S \neq P_3$. Then G being $\{R, S\}$-o_1-heavy implies G is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or Z_2.*

It is worth noticing that if a graph S contains an induced path with six or more vertices, then every graph of order n belonging to the family $\mathcal{F}(S, n)$ is S-o-heavy. If S is a P_6-free graph, then the connections between these types of heaviness do not follow any general rule. Consider for example the graph F_{4r}. For $S \in \{P_4, P_5, Z_1, Z_2\}$, F_{4r} belongs to the family $\mathcal{F}(S, n)$ but is not S-o-heavy. On the other hand, the graph G_2 depicted in Figure 2.3 is both P_4- and P_5-o_1-heavy, but it is a member of neither $\mathcal{F}(P_4, n)$ nor $\mathcal{F}(P_5, n)$.

1.4 Clique-heavy subgraphs

Recently ([34]) Li and Ning introduced another type of heavy graphs. Their motivation was the following theorem by Hu.

Theorem 1.30 (Hu [32]). *Let G be a 2-connected graph of order $n \geq 3$. If $G \in \mathcal{F}(K_{1,3}, n)$ and every induced P_4 in an induced N of G contains a vertex of degree at least $n/2$, then G is hamiltonian.*

Definition 3. Induced subgraph S of a simple graph G is *c-heavy* in G, if for every maximal clique C of S every non-trivial component of $S - C$ contains a vertex of degree at least $n/2$ in G. Graph G is said to be *S-c-heavy* if every induced subgraph of G isomorphic to S is c-heavy in G.

This notion allows to present the result by Hu in the following, simpler way.

Theorem 1.30 (Hu [32]). *Let G be a 2-connected graph of order $n \geq 3$. If $G \in \mathcal{F}(K_{1,3}, n)$ and G is N-c-heavy, then G is hamiltonian.*

Note that, in general case, properties of being c-heavy and o-heavy are independent, in the sense that none of them implies another. Consider again the Fan's graph F_{4r} represented on Fig. 2.1. One can check that this graph is N-c-heavy but not N-o-heavy. On the other hand, graph G_2 from Fig. 2.3 is both P_5-c-heavy and P_5-o-heavy but it does not belong to the family $\mathcal{F}(P_5, n)$.

Furthermore, there is no point in examining claw-c-heavy or P_3-c-heavy graphs, as the notion is in this case meaningless (every component of the claw or P_3 lacking maximal clique is trivial). Keeping that in mind the authors of [34] extended Theorem 1.10 in the following way.

Theorem 1.31 (B. Li, Ning [34]). *Let S be a connected graph with $S \neq P_3$ and let G be a 2-connected, claw-o-heavy graph. Then G being S-c-heavy implies G is hamiltonian if and only if $S = P_4, P_5, Z_1, Z_2, B, N$ or W.*

Motivated by Theorems 1.28 and 1.29 we naturally propose the notion of c_1-heaviness.

Definition 4. Induced subgraph S of G is *c_1-heavy* in G, if for every maximal clique C of S every non-trivial component of $S - C$ contains a vertex of order at least $(n+1)/2$. Graph G is called *S-c_1-heavy* if every induced subgraph of G isomorphic to S is c_1-heavy in G.

Similarly to Theorems 1.22 and 1.29, we extended Bedrossian's Theorem 1.11 in the following way.

Theorem 1.32 (WW [48]). *Let G be a 2-connected graph which is not a cycle and let S be a connected graph other than the path P_3. Then G being claw-o_1-heavy and S-c_1-heavy implies G is pancyclic if and only if $S = P_4, P_5, Z_1$ or Z_2.*

In Chapter 2 we introduce notation used further in the thesis and present some preliminary results as well as some auxiliary lemmas. Proofs of Theorems 1.19 and 1.20 are presented in Chapters 3 and 4, respectively. Chapter 5 is devoted to the proof of Theorem 1.27 and the proof of Theorem 1.32 can be found in Chapter 6.

2 Preliminaries

For a vertex $v \in V(G)$, we denote by $N_G(v)$ the neighbourhood of v, i.e., the set of vertices adjacent to v. For $A \subseteq V(G)$, we denote by $G[A]$ the subgraph of G induced by the vertex set A. The *neighbourhood of* v *in* $G[A]$, namely $N_G(v) \cap A$, is denoted by $N_A(v)$ and the *closed neighbourhood of* v *in* $G[A]$, namely $N_A(v) \cup \{v\}$, is denoted by $N_A[v]$.

For a cycle $C = v_1v_2...v_pv_1$ we distinguish one of the two possible orientations of C. We write $v_iC^+v_j$ for the path following the orientation of C, i.e., the path $v_iv_{i+1}...v_{j-1}v_j$, and $v_iC^-v_j$ denotes the path from v_i to v_j opposite to the direction of C, that is the path $v_iv_{i-1}...v_{j+1}v_j$. By $d_C(v_i, v_j)$ we denote the length of the shorter of the paths $v_iC^+v_j$ and $v_iC^-v_j$. Similarly, for a path $P = v_1...v_m$ and two vertices $v_i, v_j \in V(P)$ with $i < j$, we write $v_iP^+v_j$ for the path $v_iv_{i+1}...v_{j-1}v_j$ and $v_jP^-v_i$ for the path $v_jv_{j-1}...v_{i+1}v_i$. For two positive integers k and m satisfying $k \leq m$, we say that G contains $[k, m]$-cycles if there are cycles $C_k, C_{k+1}, ..., C_m$ in G.

Let G be a graph of order n. Vertex $v \in V(G)$ is called *heavy* if $d_G(v) \geq n/2$ and *super-heavy* if $d_G(v) \geq (n+1)/2$.

Let $A, B \subset V(G)$ be subsets of vertices of G. By $e(A, B) = |\{e = uv \in E(G)\colon u \in A, v \in B\}|$ we denote the total number of edges between A and B. If both A and B consist of one element, say $A = \{v_A\}$ and $B = \{v_B\}$, we write $e(v_A, v_B)$ instead of $e(\{v_A\}, \{v_B\})$.

The following lemma, which was listed as an exercise in [8] and proved in [4], proved to be a useful tool in working with heavy subgraphs of various types.

Lemma 2.1 (Benhocine and Wojda [4])**.** *If a graph G of order $n \geq 4$ has a cycle C of length $n-1$, such that the vertex not in $V(C)$ has degree at least $n/2$, then G is pancyclic.*

This result can be extended as follows.

Lemma 2.2 (WW [52])**.** *Let G be a graph of order $n \geq 4$ and let C be a cycle of length $n-i$ in G, for some $i \in \{1, ..., n-3\}$. If there is a vertex $v \in V(G) \setminus V(C)$ with $d_G(v) \geq (n+i-1)/2$, then there are $[3, n-i+1]$-cycles in G.*

Proof. Let $C = v_0...v_{n-i-1}v_1$ and let v be a vertex of degree at least $(n+i-1)/2$ such that $v \notin V(C)$. Let $G' = G[V(C)]$. Suppose that the statement is not true, i.e., that there is no cycle C_p in G for some $p \in \{3, ..., n-i+1\}$. Then

$$e(v, v_j) + e(v, v_{j+p-2}) \leq 1$$

for $j = 0, ..., n-i-1$, with addition of indices performed modulo $n-i$. This implies that

$$d_{G'}(v) = 1/2 \cdot \sum_{j=0}^{n-i-1} [e(v, v_j) + e(v, v_{j+p-2})] \leq (n-i)/2.$$

On the other hand, since there are $i-1$ possible neighbours of v outside the cycle C, we have

$$d_{G'}(v) \geq (n+i-1)/2 - i + 1 = (n-i+1)/2,$$

a contradiction. □

An immediate consequence of the above lemma is the following.

Corollary 2.1. *Let G be a hamiltonian graph of order n and let $v \in V(G)$ be a super-heavy vertex. If there is a cycle C of length $n-2$ in G such that $v \notin V(C)$, then G is pancyclic.*

Proof. Lemma 2.2 implies that there are $[3,\, n-1]$-cycles in G. Since G is hamiltonian, it is pancyclic. $\square$

The next four lemmas provide a description of the cycle structure of hamiltonian graphs with two vertices that lie close (i.e., with distance one or two along the cycle) to each other on some hamiltonian cycle and have large degree sum.

Lemma 2.3 (Bondy [6])**.** *Let G be a graph of order n with a hamiltonian cycle C. If there are two vertices $x,\, y \in V(G)$ such that $d_C(x,\, y) = 1$ and $d_G(x) + d_G(y) \geq n+1$, then G is pancyclic.*

Lemma 2.4 (Schmeichel and Hakimi [45])**.** *Let G be a graph of order n with a hamiltonian cycle $C = v_1v_2...v_nv_1$. If $d_G(v_1) + d_G(v_n) \geq n$, then G is pancyclic unless G is bipartite or else G is missing only the $(n-1)$-cycle.*

Furthermore, when G is missing only the $(n-1)$-cycle and $d_G(v_1) = d_G(v_2) = n/2$, then the adjacency structure near v_1 and v_2 is the following: the path $v_{n-2}v_{n-1}v_nv_1v_2v_3$ is an induced one, and v_nv_{n-3}, v_nv_{n-4}, v_1v_4, v_1v_5 are edges in G.

Lemma 2.5 (Ferrara, Jacobson and Harris [23])**.** *Let G be a graph of order n with a hamiltonian cycle C. If there are two vertices $x,\, y \in V(G)$ such that $d_C(x,\, y) = 2$ and $d_G(x) + d_G(y) \geq n+1$, then G is pancyclic.*

Lemma 2.6 (Han [31])**.** *Let G be a graph of order n with a hamiltonian cycle C. If there are two non-adjacent vertices $x,\, y \in V(G)$ such that $d_C(x,\, y) = 2$ and $d_G(x) + d_G(y) \geq n$, then G is pancyclic, unless G is bipartite or else G is missing only the $(n-1)$-cycle, or the cycle of length three.*

The next lemma will be used to derive Lemma 2.8 - a cycle structure theorem similar to Lemmas 2.3-2.6.

Lemma 2.7 (Faudree, Favaron, Flandrin and Li [18])**.** *Let $P = v_1...v_n$ be a hamiltonian path of G. If $v_1v_n \notin E(G)$ and $d_G(v_1) + d_G(v_n) \geq n$, then G is pancyclic.*

In [23] the authors prove results similar to Lemma 2.5 for pairs of vertices that lie further from each other on a hamiltonian cycle and have larger sums of degrees. The following lemma provides a more precise description of the cycle structure in the specific case that we are interested in.

Lemma 2.8 (WW)**.** *Let G be a graph with n vertices and a hamiltonian cycle C. Let $x,\, y \in V(G)$ satisfy $d_C(x,\, y) = 3$ and $d_G(x) + d_G(y) \geq n+1$, with x preceding y on C. Then (i) if $\{x,\, x^+,\, y^-,\, y\}$ induces a path or a cycle, then G is pancyclic or else missing only the $(n-1)$-cycle,*

(ii) if $\{x,\, x^+,\, y^-,\, y\}$ *induces* Z_1*, then* G *is pancyclic,*
(iii) if $xy \in E(G)$ *and* $\{x,\, x^+,\, y^-,\, y\}$ *induces* $K_4 - e$*, then* G *is pancyclic,*
(iv) if $xy \notin E(G)$ *and* $\{x,\, x^+,\, y^-,\, y\}$ *induces* $K_4 - e$*, then* G *is pancyclic or else* $d_G(x) + d_G(y) = n+1$ *and* G *is missing only the* $(n-2)$*-cycle.*

Proof. Suppose that the path xx^+y^-y is induced in G. Then the path yC^+x is a hamiltonian path in $G' = G - \{x^+,\, y^-\}$. Since $d_{G'}(x) + d_{G'}(y) \geq n-1$, it follows from Lemma 2.7 that G' is pancyclic.

If $\{x,\, x^+,\, y^-,\, y\}$ induces a cycle, then G' is hamiltonian, with the cycle xyC^+x being its Hamilton cycle. Now the pancyclicity of G' follows from Lemma 2.3. Hence, there are $[3,\, n-2]$-cycles in G. Since G is hamiltonian, this proves (i).

Now suppose that the set $\{x,\, x^+,\, y^-,\, y\}$ induces Z_1. Note that this implies that there is a cycle of length $n-1$ in G. Consider again the graph $G' = G - \{x^+,\, y^-\}$. Since $d_{G'}(x) + d_{G'}(y) \geq n-2$, the path yC^+x is a hamiltonian path in G' and xy is not an edge in G, it follows from Lemma 2.7 that G' is pancyclic. Thus (ii) holds.

Under the assumptions of (iii) exactly one of the edges xy^- and x^+y is missing in G. If $xy^- \notin E(G)$, then set $G' = G - y^-$. Otherwise let $G' = G - x^+$. In either case G' is a hamiltonian graph with a Hamilton cycle C' such that $d_{C'}(x,\, y) = 2$. Since $d_{G'}(x) + d_{G'}(y) \geq n$, it follows from Lemma 2.3 that G' is pancyclic. Pancyclicity of G' implies pancyclicity of G.

Finally, assume that the vertices from the set $\{x,\, x^+,\, y^-,\, y\}$ induce $K_4 - e$ with x and y being non-adjacent. If $d_G(x) + d_G(y) > n+1$, then the graph $G' = G - x^+$ with a hamiltonian cycle $C' = xy^-yC^+x$ is pancyclic by Lemma 2.5. This implies pancyclicity of G. Now assume $d_G(x) + d_G(y) = n+1$. Note that this implies that at least one of the vertices x and y has at least $(n+1)/2$ neighbours in G. Without loss of generality assume $d_G(x) \geq (n+1)/2$. Again, consider $G' = G - x^+$. Since now $d_{G'}(x) + d_{G'}(y) = n-1$, it follows from Lemma 2.6 that G' is pancyclic, unless it is bipartite or else missing a cycle C_3 or a cycle C_{n-2}.

Suppose that G' is missing a cycle of length three. Consider now the path $P = y^+C^+x^-$. Clearly, $d_P(x) \geq (n+1)/2 - 2 = (|V(P)| + 1)/2$. Since x can not be adjacent to two consecutive vertices of P, it follows that $|V(P)|$ is odd and x is adjacent to every second vertex of P, beginning with y^+, i.e., $N_P(x) = \{y^+,\, y^{+++},\, ...,\, x^{---},\, x^-\}$. It follows that the set $\{xC^+vx\colon v \in N_P(x)\}$ consists of cycles in G of all possible odd lengths greater than five. Similarly, for cycles of all even lengths take the set $\{xy^-C^+vx\colon v \in N_P(x)\}$. Since xx^+y^-x is a triangle in G, this implies that G is pancyclic.

Now suppose that G' contains a cycle of length three. Clearly, G' is not bipartite. By Lemma 2.6 G' is pancyclic or missing only $(n-2)$-cycle. Thus the same is true for G, since it contains a cycle xy^-C^+x of length $n-1$ and a hamiltonian cycle. The proof of (iv) is complete. □

Note that Lemma 2.8 does not provide information on the case when the set $\{x,\, x^+,\, y^-,\, y\}$ induces a complete graph. It seems that the description of the cycle structure of G in this case is not as straightforward as in the other cases.

Lemma 2.9 (WW [52]). *Let G be a graph of order n. Let u, $v \in V(G)$ and let i be some non-negative integer less than $n-1$. Let X be a set of i vertices $\{x_1, ..., x_i\} \subset V(G)$ such that $(N[u] \cup N[v]) \cap X = \emptyset$. Suppose that there are $[n-i+1, n]$ cycles in G and $G' = G - X$ is hamiltonian with a Hamilton cycle C. Then*

1. *if $d_C(u, v) \leq 2$ and $d_G(u) + d_G(v) \geq n - i + 1$, then G is pancyclic,*
2. *if $d_C(u, v) = 1$, $d_G(u) + d_G(v) \geq n - i$ and there is a $(|G'| - 1)$-cycle in G', then G is pancyclic.*

Proof. The first statement is true, since under these assumptions G' is pancyclic by Lemma 2.3 or 2.5. If the second case occurs, G' is pancyclic by Lemma 2.4. Pancyclicity of G' implies pancyclicity of G. □

We close this section with introducing notation regarding some of the special graphs appearing throughout the rest of the thesis (recall that some of them are represented on Fig. 2.2 on page 6). We say that a set of vertices $A = \{v_1, v_2, ..., v_i\} \subset V(G)$ induces a path P_i in G, if the subgraph of G induced by A is a path P_i, with its edges being v_1v_2, v_2v_3, ..., $v_{i-1}v_i$. If $A = \{v_1, v_2, v_3, v_4\}$ and $G[A]$ is isomorphic to $K_{1,3}$ with v_1v_2, v_1v_3, v_1v_4 being the edges of this claw, we say that $\{v_1; v_2, v_3, v_4\}$ induces $K_{1,3}$ (or induces a claw).

Let $A = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. If A induces D in G, with $\{v_1, v_2, v_3\}$ inducing a triangle and $\{v_5, v_4, v_2, v_3, v_6, v_7\}$ inducing a path, we say that $\{v_1, v_2, v_3; v_4, v_5; v_6, v_7\}$ induces a D (or induces a deer).

Finally, let $A = \{v_1, v_2, v_3, v_4, v_5\}$. If $G[A]$ is isomorphic to H, with v_1 being the vertex of degree four in H, and the only edges of H not containing v_1 being v_2v_3 and v_4v_5, we say that $\{v_1; v_2, v_3; v_4, v_5\}$ induces an H.

3 Proof of Theorem 1.19

The basic tool applied in the proof of Theorem 1.19 is the following result, stated implicitly in [5].

Theorem 3.1 (Bondy [5]). *Let G be a 2-connected graph of order $|V(G)| \geq k$ and let $P = v_1...v_m$ be a path of maximum length in G. If $d_G(v_1) + d_G(v_m) \geq k$, then there is a cycle of length at least k in G.*

In [5], in the first paragraph of the proof of Theorem 1, the assumptions of Theorem 1 are used to prove the existence of a longest path satisfying the assumptions of Theorem 3.1. In the remaining part of the proof it is showed that the assumptions of Theorem 3.1 imply the existence of a cycle of length at least k.

For the convenience of the reader, we restate Theorem 1.19 below.

Theorem 1.19 (WW [51]) *Let G be a 2-connected graph with n vertices. If $3 \leq k \leq n$ and $G \in \mathcal{F}(\{K_{1,3}, P_4\}, k)$, then there is a cycle of length at least k in G.*

We use the general idea of the proof of Theorem 1.17. In our case, however, this leads to much more complex considerations. The idea is to choose a longest path in G that possesses some specific properties and to seek for a contradiction with Theorem 3.1.

Proof of Theorem 1.19: Suppose that there are no cycles of length at least k in G. It will be shown that this leads to the existence of a longest path $P = v_1...v_m$ in G such that $d_G(v_1) + d_G(v_m) \geq k$, contradicting Theorem 3.1.

For a given longest path $P = v_1...v_m$ in G let v_{l_P} be the last neighbour of v_1 along P, i.e., $l_P = \max\{i \colon v_1 v_i \in E(G)\}$, and let v_{n_P} be the last nonneighbour of v_1 preceding v_{l_P}, that is $n_P = \max\{i \colon i < l_P \text{ and } v_1 v_{n_P} \notin E(G)\}$.

Clearly, $l_P > 2$. Furthermore, it follows from 2-connectivity of G that $l_P < m$, since otherwise there would be either a hamiltonian cycle or a path longer than P in G. Next observe that there exists a longest path P with $n_P > 2$. If this is not the case and $n_P = 1$, let Q be a path from v_i to v_j, $i \leq l_P - 1$, $j \geq l_P + 1$, such that $V(P) \cap V(Q) = \{v_i, v_j\}$. Then form the path $P' = v_{j-1} P^- v_{i+1} v_1 P^+ v_i Q^+ v_j P^+ v_m$, which is a longest path with $l_{P'} \geq j > l_P$, a contradiction when P is chosen to have the largest l_P value.

Fix a longest path $P = v_1...v_m$ with n_P of largest possible value. With the above observations it will next be shown that there exists a longest path with one of its end-vertices being v_m and the other having degree at least $k/2$. To do this, suppose that $d_G(v_1) < k/2$. Note that, since $n_P > 2$, we have $d_G(v_{n_P}) < k/2$, since otherwise the path $v_{n_P} P^- v_1 v_{n_P+1} P^+ v_m$ is a longest path with $d_G(v_{n_P}) \geq k/2$. Since $G \in \mathcal{F}(K_{1,3}, k)$, it follows that $\{v_{n_P+1}; v_1, v_{n_P}, v_{n_P+2}\}$ can not induce a claw. Thus v_{n_P+2} is adjacent to at least one of the vertices v_1 and v_{n_P}. Before the proof divides into subcases, we note that $d_G(v_{n_P+1}) < k/2$, since by the previous observation at least one of the paths

$v_{n_P+1}P^-v_1v_{n_P+2}P^+v_m$ or $v_{n_P+1}v_1P^+v_{n_P}v_{n_P+2}P^+v_m$ is a longest path in G beginning with v_{n_P+1}.

Throughout the proof, whenever we declare a contradiction due to a discovered induced subgraph of G isomorphic to the claw or the path P_4, it is because the subgraph does not satisfy Fan's condition with constant k.

Case 1: $v_1v_{n_P+2} \in E(G)$, $v_{n_P}v_{n_P+2} \notin E(G)$

Note that under the assumptions of this case we have $m \geq n_P+3$. We begin with crucial pieces of information regarding the degree of the vertex v_{n_P+2} and the adjacency structure of its neighbourhood.

Claim 3.1. *$v_{n_P+3}v_{n_P}$, $v_{n_P+3}v_{n_P+1} \notin E(G)$ and $d_G(v_{n_P+2}) \geq k/2$.*

Proof. Note that if v_{n_P+3} is adjacent to v_{n_P}, then under the assumptions of this case the path $P' = v_{n_P}P^-v_1v_{n_P+1}P^+v_m$ is a longest path in G with $n_{P'} \geq n_P+2$, contradicting the choice of P. Similarly, if $v_{n_P+3}v_{n_P+1} \in E(G)$, then $P' = v_{n_P}P^-v_1v_{n_P+2}v_{n_P+1}v_{n_P+3}P^+v_m$ is a longest path with $n_{P'} \geq n_P+1$. Thus v_{n_P+3} is adjacent neither to v_{n_P}, nor to v_{n_P+1}, and so $v_{n_P}v_{n_P+1}v_{n_P+2}v_{n_P+3}$ is an induced path P_4 in G. Since $G \in \mathcal{F}(P_4, k)$ and $d_G(v_{n_P}) < k/2$, it follows that $d_G(v_{n_P+2}) \geq k/2$. □

Claim 3.2. *$d_G(v_2) < k/2$ and $v_2v_{n_P+3} \notin E(G)$, $v_2v_{n_P+1} \in E(G)$.*

Proof. Clearly, if $d_G(v_2) \geq k/2$, then $v_2P^+v_{n_P+1}v_1v_{n_P+2}P^+v_m$ is a longest path with $d_G(v_2) \geq k/2$, and if $v_2v_{n_P+3} \in E(G)$, then, by Claim 3.1, $v_{n_P+2}v_1v_{n_P+1}P^-v_2v_{n_P+3}P^+v_m$ is a longest path $d_G(v_{n_P+2}) \geq k/2$.

Now suppose that v_2 is not adjacent to v_{n_P+1}. Since $d_G(v_2)$, $d_G(v_{n_P+1}) < k/2$ and $G \in \mathcal{F}(\{K_{1,3}, P_4\}, k)$, the set $\{v_2, v_1, v_{n_P+1}, v_{n_P}\}$ can not induce P_4 and the set $\{v_{n_P+2}; v_2, v_{n_P+1}, v_{n_P+3}\}$ can not induce a claw. It follows from Claim 3.1 that $v_2v_{n_P} \in E(G)$ and $v_2v_{n_P+2} \notin E(G)$. But now $v_2v_{n_P}v_{n_P+1}v_{n_P+2}$ is an induced P_4 in G, a contradiction. □

Claim 3.3. *There are no edges between the vertices v_1, v_{n_P}, v_{n_P+1} and the vertices from the set $\{v_{n_P+3}, ..., v_m\}$.*

Proof. From the definition of n_P it follows that to prove that v_1 is not adjacent to any of the vertices from the set $\{v_{n_P+3}, ..., v_m\}$ it suffices to show that it is not adjacent to v_{n_P+3}. This is clearly true, since otherwise $v_{n_P+2}P^-v_1v_{n_P+3}P^+v_m$ would be a longest path with $d_G(v_{n_P+2}) \geq k/2$, by Claim 3.1.

Recall that $v_{n_P}v_{n_P+3} \notin E(G)$ and $v_{n_P+1}v_{n_P+3} \notin E(G)$, by Claim 3.1, and $v_2v_{n_P+1} \in E(G)$, by Claim 3.2. Suppose that v_{n_P} is adjacent to v_{n_P+j} for some $3 < j \leq m-n_P$. Then the path $P' = v_{n_P}P^-v_2v_{n_P+1}v_1v_{n_P+2}P^+v_m$ is a longest path in G with $n_{P'} \geq n_P+3$, contradicting the choice of P.

From the observations made so far it follows that if v_{n_P+1} is adjacent to some vertex v_{n_P+j} with $3 < j \leq m-n_P$, then $\{v_{n_P+1}; v_1, v_{n_P}, v_{n_P+j}\}$ induces a claw. Since $d_G(v_1)$, $d_G(v_{n_P}) < k/2$, this contradicts G being a graph from the family $\mathcal{F}(K_{1,3}, k)$. □

The next claim provides a characterization of properties of the vertices that lie on P between v_1 and v_{n_P}.

Claim 3.4. *For $i \in \{2, ..., n_P\}$ the following holds.*
(i) $d_G(v_i) < k/2$,
(ii) $v_i v_{n_P+3} \notin E(G)$,
(iii) $v_i v_{n_P+1} \in E(G)$,
(iv) either v_i is adjacent to both v_1 and v_{n_P+2} or else it is not adjacent to any of them,
(v) v_i is not adjacent to any of the vertices from the set $\{v_{n_P+3}, ..., v_m\}$.

Proof. The proof is by induction on i. For $i = 2$ the statements (i), (ii) and (iii) are true by Claim 3.2. To show that the condition (iv) holds, we first observe that v_2 is adjacent to v_1. Suppose $v_2 v_{n_P+2} \notin E(G)$. Then under the assumptions of the case and depending on the existence of the edge $v_2 v_{n_P}$, either $v_{n_P} v_2 v_1 v_{n_P+2}$ is an induced path or the set $\{v_{n_P+1}; v_2, v_{n_P}, v_{n_P+2}\}$ induces a claw. Since the degrees of v_1, v_2 and v_{n_P} are strictly less than $k/2$, this contradicts G being a member of the family $\mathcal{F}(\{K_{1,3}, P_4\}, k)$.

For the proof of (v) suppose that v_2 is adjacent to some vertex $v \in \{v_{n_P+3}, ..., v_m\}$. The path $v v_2 v_{n_P+1} v_{n_P}$ can not be an induced one, since $d_G(v_2)$, $d_G(v_{n_P}) < k/2$. Thus it follows from Claim 3.3 that v_2 is adjacent to v_{n_P}. But now $\{v_2; v, v_{n_P}, v_1\}$ induces a claw with $d_G(v_2)$, $d_G(v_{n_P}) < k/2$, a contradiction.

Now assume that for some $i < n_P$ the conditions (i)-(v) hold for the vertices $v_2, ..., v_i$. It will be shown that they hold also for v_{i+1}.

First observe that $d_G(v_{i+1}) < k/2$, since otherwise, by the condition (iii) for v_i, the path $v_{i+1} P^+ v_{n_P+1} v_i P^- v_1 v_{n_P+2} P^+ v_m$ is a longest path in G with its first vertex having degree at least $k/2$. The validity of the condition (ii) is also straightforward: if $v_{i+1} v_{n_P+3} \in E(G)$, then a longest path with its first vertex having degree not less than $k/2$ is the path $v_{n_P+2} v_1 P^+ v_i v_{n_P+1} P^- v_{i+1} v_{n_P+3} P^+ v_m$, by Claim 3.1.

Now suppose that the condition (iii) is not true, i.e., that $v_{i+1} v_{n_P+1}$ is not an edge in G. It follows that v_{i+1} is not adjacent to v_{n_P+2}, since otherwise, by (ii) for v_{i+1} and by Claim 3.3, the set $\{v_{n_P+2}; v_{i+1}, v_{n_P+1}, v_{n_P+3}\}$ induces a claw with $d_G(v_{i+1})$, $d_G(v_{n_P+1}) < k/2$.

If $v_i v_{n_P}$ is not an edge in G, then by (iii) for v_i, the vertex v_{i+1} is adjacent to v_{n_P} in order to avoid induced path $v_{i+1} v_i v_{n_P+1} v_{n_P}$ with $d_G(v_i)$, $d_G(v_{n_P}) < k/2$. But now $v_{i+1} v_{n_P} v_{n_P+1} v_{n_P+2}$ is an induced P_4 with none of the vertices v_{i+1} and v_{n_P+1} having degree not less than $k/2$, a contradiction. Hence, $v_i v_{n_P} \in E(G)$.

Note that v_i can not be adjacent to v_{n_P+2}. If this is not the case, then, depending on the existence of the edge $v_{i+1} v_{n_P}$, either $\{v_i; v_{n_P}, v_{n_P+2}, v_{i+1}\}$ is an induced claw in G or else $v_{i+1} v_{n_P} v_{n_P+1} v_{n_P+2}$ is an induced path P_4 that does not satisfy the Fan's condition.

From the fact that $v_i v_{n_P+2}$ is not an edge in G and from the condition (iv) for v_i it follows that $v_i v_1 \notin E(G)$. This implies that v_{i+1} is adjacent to v_1, since otherwise the path $v_{i+1} v_i v_{n_P+1} v_1$ is an induced P_4 with $d_G(v_1)$, $d_G(v_i) < k/2$. But now $v_i v_{i+1} v_1 v_{n_P+2}$ is an induced P_4 with $d_G(v_1)$, $d_G(v_i) < k/2$, a contradiction. Thus the condition (iii) holds for v_{i+1}.

To show that the condition (iv) is satisfied by v_{i+1}, first suppose that $v_{i+1}v_{n_P} \notin E(G)$. Then v_{i+1} is adjacent to both v_1 and v_{n_P+2} to avoid induced claws $\{v_{n_P+1};\, v_{i+1},\, v_{n_P},\, v_1\}$ and $\{v_{n_P+1};\, v_{i+1},\, v_{n_P},\, v_{n_P+2}\}$ with both v_{i+1} and v_{n_P} having degrees less than $k/2$.

Now suppose that v_{i+1} is adjacent to v_{n_P}. If v_1 is a neighbour of v_{i+1}, then the same is true for v_{n_P+2}, since otherwise $v_{n_P}v_{i+1}v_1v_{n_P+2}$ is an induced P_4 with $d_G(v_1),\, d_G(v_{n_P}) < k/2$. Similarly, $v_{i+1}v_{n_P+2} \in E(G)$ implies that v_{i+1} is adjacent to v_1, to avoid induced path $v_{n_P}v_{i+1}v_{n_P+2}v_1$. This proves (iv).

Finally, suppose that v_{i+1} is adjacent to some vertex $v \in \{v_{n_P+3},\, ...,\, v_m\}$. By Claim 3.3 we can assume that $i+1 < n_P$. If $v_{i+1}v_1 \notin E(G)$, then $v_1v_{n_P+1}v_{i+1}v$ is an induced path P_4, by Claim 3.3. Since the degrees of both v_1 and v_{i+1} are less than $k/2$, this contradicts G belonging to the family $\mathcal{F}(P_4,\, k)$. Now suppose that v_{i+1} is adjacent to v_1. Then $v_{i+1}v_{n_P} \notin E(G)$ to avoid induced claw $\{v_{i+1};\, v_1,\, v_{n_P},\, v\}$. But now $v_{n_P}v_{n_P+1}v_{i+1}v$ is an induced path P_4, by Claim 3.3. This final contradiction shows that the property (v) holds for v_{i+1}. By mathematical induction the claim is true. □

Claim 3.5. *For every $i \in \{1,\, ...,\, n_P+1\}$ the neighbourhood $N_G(v_i)$ of the vertex v_i is a subset of the set $\{v_1,\, v_2,\, ...,\, v_{n_P+2}\}$.*

Proof. Note that by Claims 3.3 and 3.4 the vertex v_i, with $1 \le i \le n_P+1$, has no neighbours in the set $\{v_{n_P+3},\, ...,\, v_m\}$. Thus to prove the claim it suffices to show that v_i is not adjacent to any $v \in V(G) \setminus V(P)$. Clearly, if one of the vertices v_1, v_2 and v_{n_P+1} was adjacent to some vertex $v \notin V(P)$, this would create a path in G longer than P, i.e., one of the paths $vv_1P^+v_m$, $vv_2P^+v_{n_P+1}v_1v_{n_P+2}P^+v_m$ or $vv_{n_P+1}P^-v_1v_{n_P+2}P^+v_m$. Hence, the claim is true for $i \in \{1,\, 2,\, n_P+1\}$.

For a proof by induction assume that the claim holds for the values from the set $\{1,\, 2,\, ...,\, i\}$, where $2 \le i \le n_P - 1$. It will be shown that this implies the validity of the claim for $i+1$.

Suppose that there is a vertex $v \in V(G) \setminus V(P)$ adjacent to v_{i+1}. Then v is not adjacent to any of v_i and v_{i+2}, since such an edge would create a path in G longer than P. Recall that $d_G(v_i),\, d_G(v_{i+2}) < k/2$, by Claim 3.4, and so $\{v_{i+1};\, v_i,\, v,\, v_{i+2}\}$ can not induce a claw in G. Thus $v_iv_{i+2} \in E(G)$. We observe that if v_{i+1} is not adjacent to some vertex v_k with $1 \le k \le i-1$, then choosing k of largest possible value gives an induced path $v_kv_{k+1}v_{i+1}v$, by the induction hypothesis. This contradicts G being a member of the family $\mathcal{F}(P_4,\, k)$, by Claim 3.4. Thus v_{i+1} is adjacent to every vertex preceding it on the path P, in particular $v_1v_{i+1} \in E(G)$. But now $vv_{i+1}v_1P^+v_iv_{i+2}P^+v_m$ is a path longer than P, a contradiction.

□

Now it follows from Claim 3.5 that $G - v_{n_P+2}$ is not connected. This contradicts G being 2-connected and completes the proof of this case.

Case 2: $v_1v_{n_P+2} \notin E(G)$, $v_{n_P}v_{n_P+2} \in E(G)$

We begin the proof of this case with a counterpart of Claim 3.3.

Claim 3.6. *There are no edges between the vertices v_1, v_{n_P}, v_{n_P+1} and the vertices from the set $\{v_{n_P+3}, ..., v_m\}$.*

Proof. The validity of the claim for v_1 follows immediately from the definition of n_P and the assumptions of this case. For v_{n_P+1} we first observe that $v_{n_P+1}v_{n_P+3} \notin E(G)$, since otherwise the path $P' = v_{n_P+2}v_{n_P}P^-v_1v_{n_P+1}v_{n_P+3}P^+v_m$ is a longest path in G with $n_{P'} \geq n_P + 1$, contradicting the choice of P. With this observation it is easy to see that if $v_{n_P+1}v \in E(G)$ for some vertex $v \in \{v_{n_P+4}, ..., v_m\}$, then the path $P' = v_{n_P+1}v_1P^+v_{n_P}v_{n_P+2}P^+v_m$ is a longest path with $n_{P'} \geq n_P + 3$. Finally, if v_{n_P} has a neighbour in the set $\{v_{n_P+3}, ..., v_m\}$, say v, then $v_1v_{n_P+1}v_{n_P}v$ is an induced P_4 in G with $d_G(v_{n_P})$, $d_G(v_1) < k/2$. A contradiction. □

Next we establish some properties of the vertices that preceed v_{n_P} on P.

Claim 3.7. *For $i \in \{1, 2, ..., n_P - 2\}$ the following holds.*
(i) $d_G(v_{n_P-i}) < k/2$,
(ii) v_{n_P-i} is adjacent to at least one of the vertices v_1 and v_{n_P+1},
(iii) $v_{n_P-i}v_{n_P+2} \in E(G)$ or else v_{n_P-i} is adjacent to v_1,
(iv) v_{n_P-i} is not adjacent to any of the vertices from the set $\{v_{n_P+3}, ..., v_m\}$.

Proof. We use induction on i. For $i = 1$ it is clear that $d_G(v_{n_P-1}) < k/2$, since the path $v_{n_P-1}P^-v_1v_{n_P+1}v_{n_P}v_{n_P+2}P^+v_m$ is a longest path in G beginning with v_{n_P-1}. Thus (i) holds. Recall that the degrees of both v_1 and v_{n_P} are less than $k/2$, and so the path $v_{n_P-1}v_{n_P}v_{n_P+1}v_1$ can not be an induced one. This implies (ii).

To show (iii) assume that v_{n_P-1} is not adjacent to v_{n_P+2} and suppose $v_1v_{n_P-1} \notin E(G)$. Then v_{n_P-1} is adjacent to v_{n_P+1} by (ii). But this implies that $\{v_{n_P+1}; v_1, v_{n_P-1}, v_{n_P+2}\}$ induces a claw. By (i), this contradicts G belonging to the family $\mathcal{F}(K_{1,3}, k)$.

For the proof of (iv) suppose that v_{n_P-1} has a neighbour, say v, in the set $\{v_{n_P+3}, ..., v_m\}$. Then v_{n_P-1} is not adjacent to v_1, since otherwise $\{v_{n_P-1}; v_1, v_{n_P}, v\}$ induces a claw, by Claim 3.6. It follows from (ii) that $v_{n_P-1}v_{n_P+1} \in E(G)$. But now $v_1v_{n_P+1}v_{n_P-1}v$ is an induced path P_4 with $d_G(v_1)$, $d_G(v_{n_P-1}) < k/2$, a contradiction. This proves (iv) for $i = 1$.

Now assume that the claim holds for the values from the set $\{1, 2, ..., i\}$, where $1 \leq i \leq n_P - 3$. It will be shown that this implies the validity of the claim for $i + 1$.

By the condition (iii) for v_{n_P-i} there is a longest path in G beginning with v_{n_P-i-1}, namely $v_{n_P-i-1}P^-v_1v_{n_P+1}P^-v_{n_P-i}v_{n_P+2}P^+v_m$ or $v_{n_P-i-1}P^-v_1v_{n_P-i}P^+v_m$. Thus $d_G(v_{n_P-i-1}) < k/2$, proving (i). For the proof of (ii) suppose that v_{n_P-i-1} is not adjacent neither to v_1 nor to v_{n_P+1}. This implies that both v_1 and v_{n_P+1} are neighbours of v_{n_P-i}, since otherwise, by (ii), one of the paths $v_{n_P-i-1}v_{n_P-i}v_1v_{n_P+1}$ and $v_{n_P-i-1}v_{n_P-i}v_{n_P+1}v_1$ would be an induced P_4 in G. Furthermore, v_{n_P} is not adjacent to v_{n_P-i-1}, to avoid induced path $v_{n_P-i-1}v_{n_P}v_{n_P+1}v_1$. It is also not adjacent to v_{n_P-i}, since otherwise $\{v_{n_P-i}; v_1, v_{n_P-i-1}, v_{n_P}\}$ induces a claw. But now $v_{n_P-i-1}v_{n_P-i}v_{n_P+1}v_{n_P}$ is an induced path with four vertices. Since the degrees of the vertices of this path are less than $k/2$, this contradicts G belonging to the family $\mathcal{F}(P_4, k)$ and proves (ii).

Now assume $v_{n_P-i-1}v_{n_P+2} \notin E(G)$ and suppose that v_{n_P-i-1} is not adjacent to v_1. From the condition (ii) for v_{n_P-i-1} it follows that $\{v_{n_P+1}; v_1, v_{n_P-i-1}, v_{n_P+2}\}$ induces a claw. Since the degrees of both v_1 and v_{n_P-i-1} are strictly less than $k/2$, this is a contradiction. Thus (iii) holds.

Finally, suppose that v_{n_P-i-1} is adjacent to some vertex $v \in \{v_{n_P+3}, ..., v_m\}$. Claim 3.6 implies that $n_P - i - 1 > 1$. If $v_{n_P-i-1}v_1 \notin E(G)$, then it follows from the condition (ii) and Claim 3.6 that the path $v_1v_{n_P+1}v_{n_P-i-1}v$ is an induced P_4 in G with the degrees of both v_1 and v_{n_P-i-1} being less than $k/2$. Thus $v_{n_P-i-1}v_1 \in E(G)$. This implies that v_{n_P-i-1} is not adjacent to v_{n_P}, since otherwise $\{v_{n_P-i-1}; v_1, v_{n_P}, v\}$ induces a claw, by Claim 3.6. Furthermore, in order to avoid induced path $v_{n_P}v_{n_P+1}v_{n_P-i-1}v$, the vertex v_{n_P-i-1} can not be adjacent to v_{n_P+1}. But now we obtain an induced path $v_{n_P-i-1}v_1v_{n_P+1}v_{n_P}$, a contradiction. By mathematical induction the claim is true. □

Claim 3.8. *For every $i \in \{0, 1, ..., n_P\}$ the neighbourhood $N_G(v_{n_P-i+1})$ of the vertex v_{n_P-i+1} is a subset of the set $\{v_1, ..., v_{n_P+2}\}$.*

Proof. Note that by Claims 3.6 and 3.7 the vertex v_{n_P-i+1}, with $0 \le i \le n_P$, has no neighbours in the set $\{v_{n_P+3}, ..., v_m\}$. Thus to prove the claim it suffices to show that v_{n_P-i+1} is not adjacent to any $v \in V(G) \setminus V(P)$. Clearly, if one of the vertices v_1, v_{n_P} and v_{n_P+1} was adjacent to some vertex v lying outside the path P, this would create a path in G longer than P, i.e., one of the paths $vv_1P^+v_m$, $vv_{n_P}P^-v_1v_{n_P+1}P^+v_m$ or $vv_{n_P+1}v_1P^+v_{n_P}v_{n_P+2}P^+v_m$. Hence, the claim is true for $i \in \{0, 1, n_P\}$.

For a proof by induction assume that the claim holds for the values from the set $\{0, 1, ..., i\}$, where $1 \le i \le n_P - 2$. It will be shown that this implies the validity of the claim for $i + 1$.

Suppose that there is a vertex $v \in V(G) \setminus V(P)$ adjacent to v_{n_P-i}. Then v is not adjacent to any of v_{n_P-i-1} and v_{n_P-i+1}, to avoid creating a path in G longer than P. Recall that $d_G(v_{n_P-i-1})$, $d_G(v_{n_P-i+1}) < k/2$, by Claim 3.7 and by the fact that $d_G(v_1) < k/2$, and so $\{v_{n_P-i}; v_{n_P-i-1}, v, v_{n_P-i+1}\}$ can not induce a claw in G. Thus $v_{n_P-i-1}v_{n_P-i+1} \in E(G)$. Next we note that if v_{n_P-i} is not adjacent to some vertex v_k with $n_P - i < k \le n_P$, then choosing k of smallest possible value gives an induced path $vv_{n_P-i}v_{k-1}v_k$, by the induction hypothesis. This contradicts G being a member of the family $\mathcal{F}(P_4, k)$, by Claim 3.7 and by the fact that $d_G(v_{n_P}) < k/2$. Thus v_{n_P-i} is adjacent to every vertex from the set $\{v_{n_P-i+1}, ..., v_{n_P+1}\}$. But now the path $vv_{n_P-i}v_{n_P+1}v_1P^+v_{n_P-i-1}v_{n_P-i+1}P^+v_{n_P}v_{n_P+2}P^+v_m$ is a path longer than P, a contradiction. □

Similarly to the previous case of the proof, now it follows from Claim 3.8 that $G - v_{n_P+2}$ is not connected, a contradiction with the assumption of 2-connectivity of G.

Case 3: $v_1v_{n_P+2} \in E(G)$, $v_{n_P}v_{n_P+2} \in E(G)$

Recall that the degrees of the vertices v_1, v_{n_P} and v_{n_P+1} are less than $k/2$. Keeping that in mind, we first establish some basic facts regarding the vertex v_{n_P-1}.

Claim 3.9. *$d_G(v_{n_P-1}) < k/2$, $v_{n_P-1}v_{n_P+1} \notin E(G)$, $v_{n_P-1}v_1 \in E(G)$.*

Proof. Note that under the assumptions of this case the path $v_{n_P-1}P^-v_1v_{n_P+1}v_{n_P}v_{n_P+2}P^+v_m$ is a longest path in G. Thus $d_G(v_{n_P-1}) < k/2$. Now suppose that v_{n_P-1} is adjacent to v_{n_P+1}.

Then the path $P' = v_1P^+v_{n_P-1}v_{n_P+1}v_{n_P}v_{n_P+2}P^+v_m$ is a longest path in G with $n_{P'} = n_p+1$, contradicting the choice of P. Hence, $v_{n_P-1}v_{n_P+1} \notin E(G)$. This implies that $v_{n_P-1}v_1 \in E(G)$, since otherwise the path $v_1v_{n_P+1}v_{n_P}v_{n_P-1}$ would be an induced P_4 in G. □

Claim 3.10. *Every neighbour of v_1 in G is adjacent to at least one of the vertices v_{n_P-1} and v_{n_P+1}.*

Proof. If this is not the case, then there exists a neighbour v of v_1 such that $\{v_1;\, v_{n_P-1},\, v_{n_P+1},\, v\}$ induces a claw, by Claim 3.9. Since $d_G(v_{n_P+1}) \le k/2$ and, by Claim 3.9, $d_G(v_{n_P-1}) \le k/2$, this contradicts G belonging to the family $\mathcal{F}(K_{1,3},\, k)$. □

Now we focus our attention on the edges $v_{n_P-1}v_{n_P+2}$, $v_{n_P-1}v_{n_P+3}$ and $v_{n_P+1}v_{n_P+3}$. We begin with the following observation.

Claim 3.11. *v_{n_P+3} is adjacent to exactly one of the vertices v_{n_P-1} and v_{n_P+1}.*

Proof. Suppose the contrary. If the vertex v_{n_P+3} is not adjacent to any of the vertices v_{n_P-1}, v_{n_P+1}, then it follows from Claim 3.10 that $v_1v_{n_P+3} \notin E(G)$. Now, depending on the existence of the edge $v_{n_P}v_{n_P+3}$, we obtain induced path $v_{n_P+3}v_{n_P}v_{n_P+1}v_1$ or induced claw $\{v_{n_P+2};\, v_{n_P},\, v_1,\, v_{n_P+3}\}$, a contradiction.

If both $v_{n_P+3}v_{n_P-1}$ and $v_{n_P+3}v_{n_P+1}$ are edges in G, then the path $P' = v_{n_P-1}P^-v_1v_{n_P+2}v_{n_P}$ $v_{n_P+1}v_{n_P+3}P^+v_m$ is a longest path in G with $n_{P'} \ge n_P + 2$, by Claim 3.9. This contradicts the choice of P. □

Claim 3.12. *$v_{n_P-1}v_{n_P+3}$ is not an edge in G.*

Proof. Suppose that the opposite holds. If $v_{n_P-1}v_{n_P+2}$ is not an edge in G, then the path $P' = v_{n_P-1}P^-v_1v_{n_P+1}v_{n_P}v_{n_P+2}v_{n_P+3}P^+v_m$ is a longest path in G with $n_{P'} \ge n_P + 2$, contradicting the choice of P. Thus $v_{n_P-1}v_{n_P+2} \in E(G)$.

It follows from Claim 3.11 that v_{n_P+1} is not adjacent to v_{n_P+3}. Since $d_G(v_{n_P+1}) < k/2$ and, by Claim 3.9, $d_G(v_{n_P-1}) < k/2$, the path $v_{n_P+1}v_{n_P}v_{n_P-1}v_{n_P+3}$ can not be an induced one. Thus it follows from Claim 3.9 that $v_{n_P}v_{n_P+3}$ is an edge in G. Now to avoid induced path $v_1v_{n_P+1}v_{n_P}v_{n_P+3}$, the vertex v_1 is adjacent to v_{n_P+3}. But then the path $P' = v_1P^+v_{n_P-1}v_{n_P+2}v_{n_P+1}v_{n_P}v_{n_P+3}P^+v_m$ is a longest path in G with $n_{P'} \ge n_p + 2$, a contradiction. □

From Claims 3.11 and 3.12 it follows that the vertex v_{n_P+3} is not adjacent to v_{n_P-1} and that it is adjacent to v_{n_P+1}. Next we observe that to avoid induced path $v_{n_P-1}v_{n_P}v_{n_P+1}v_{n_P+3}$ the vertex v_{n_P+3} is adjacent to v_{n_P}, by Claim 3.9. It follows that v_{n_P+3} is adjacent also to v_1, since otherwise the path $v_1v_{n_P-1}v_{n_P}v_{n_P+3}$ is an induced one, also by Claim 3.9. But now, depending on the existence of the edge $v_{n_P-1}v_{n_P+2}$, one of the paths $P' = v_1P^+v_{n_P-1}v_{n_P+2}P^-v_{n_P}v_{n_P+3}P^+v_m$ and $P'' = v_{n_P-1}P^-v_1v_{n_P+1}v_{n_P+2}v_{n_P}v_{n_P+3}P^+v_m$ is a longest path in G. Since $n_{P'} \ge n_P + 2$ and $n_{P''} \ge n_P + 1$, this contradicts the choice of P. This final contradiction completes the proof of this case and shows that there exists a longest path in G with one if its end vertices being v_m and with the other one having degree at least $k/2$.

In the above argument, each longest path considered has v_m as one of the end vertices. Thus, since one of the end vertices of P has degree not less than $k/2$, it could have been initially assumed that P is a longest path with $d_G(v_m) \geq k/2$ and with n_P of largest possible value. The above argument then shows that there exists a longest path P with both end vertices of degree not less than $k/2$. This contradiction with Theorem 3.1 completes the proof of the theorem. $\square$

4 Proof of Theorem 1.20

We begin this Chapter with restating Theorem 1.20.

Theorem 1.20 (WW [51]) *Let G be a 2-connected graph of order $n \geq 3$. If $G \in \mathcal{F}(\{K_{1,3},\, P_4\},\, n)$, then G is pancyclic unless $n = 4r$, $r > 2$ and G is F_{4r}, or n is even and $G = K_{n/2,\,n/2}$ or else $n \geq 6$ and $G = K_{n/2,\,n/2} - e$.*

We first prove three auxiliary lemmas that deal with the exceptional non-pancyclic graphs and establish the existence of short cycles in a graph satisfying the assumptions of Theorem 1.20.

Lemma 4.1 (WW [51])**.** *Let G be a 2-connected, bipartite graph of order $n \geq 3$. If $G \in \mathcal{F}(\{K_{1,3},\, P_4\},\, n)$, then n is even and either $G = K_{n/2,\,n/2}$ or else $n \geq 6$ and $G = K_{n/2,\,n/2} - e$.*

Proof. First suppose that G is $\{K_{1,3},\, P_4\}$-free. Then it follows from Theorem 1.11 that G is a cycle. Since there are no induced paths with four vertices in G, G is a cycle $K_{2,2}$.

Now assume that G contains an induced claw or an induced path P_4. Let $(X,\, Y)$ be a bipartition of $V(G)$. It follows from the assumptions that there is a vertex, say u, in G with $d_G(u) \geq n/2$. Clearly, if $|V(G)| = 4$, then G is isomorphic to $K_{2,2}$. Since G is bipartite and, by Theorem 1.19, hamiltonian, its order n is even. Thus assume $|V(G)| \geq 6$. Without loss of generality let X be the set of bipartition containing u. It follows that $|Y| \geq n/2 \geq 3$. Note that since $G \in \mathcal{F}(K_{1,3},\, n)$ and u together with any three of its neighbours induce a claw, at most one neighbour of u has degree less than $n/2$. This implies $|X| = |Y| = n/2$. By the symmetry, at most one vertex in X might have less neighbours than $n/2$. Let $x \in X$ and $y \in Y$ be those only vertices in G, the degree of which is not necessarily equal to $n/2$. Clearly, every vertex of Y other than y is adjacent to x and every vertex from X other than x is adjacent to y. Thus, depending on the existence of the edge xy in G, G is isomorphic either to $K_{n/2,\,n/2}$ or else to $K_{n/2,\,n/2} - e$. □

Note that under additional assumption of G not being a cycle Lemma 4.1 remains valid if the pair $\{K_{1,3},\, P_4\}$ is replaced by any of the pairs of subgraphs appearing in Theorem 1.11. Similarly, it seems that the next lemma also could be adapted for these other pairs. This might be a good first step towards proving Conjecture 1.2 (proposed on page 10).

Lemma 4.2 (WW [51])**.** *Let G be a 2-connected, non-bipartite graph of order n. If $G \in \mathcal{F}(\{K_{1,3},\, P_4\},\, n)$ and there are no cycles of length $n-1$ in G, then G is isomorphic to F_{4r}, with $r > 2$.*

Proof. Suppose that G is $\{K_{1,3},\, P_4\}$-free. Similarly to the previous Lemma, this implies that G is a cycle $K_{2,2}$, by Theorem 1.11. This contradicts the assumption of G not being bipartite. Hence, we can assume that G contains an induced claw or an induced path P_4, and so there are at least two heavy vertices in G.

Note that by Theorem 1.19 G is hamiltonian. It is easy to check that if G has no more than five vertices, then it is pancyclic. Thus we assume $|V(G)| \geq 6$. Let $C = v_0 ... v_{n-1} v_0$ be a hamiltonian cycle in G. Clearly, under the assumptions of the Lemma there are no edges of the form v_iv_{i+2} in G. In the following any arithmetic involving the subscripts of the vertices of C is modulo n. We begin the proof with an observation regarding heavy vertices of G.

Claim 4.1. *If v_i is a heavy vertex in G, then at least one of the vertices v_{i-1} and v_{i+1} is also heavy.*

Proof. Suppose to the contrary that neither v_{i-1} nor v_{i+1} is heavy. Since $G \in \mathcal{F}(P_4, n)$, this implies that none of the paths $v_{i-2}v_{i-1}v_iv_{i+1}$ and $v_{i-1}v_iv_{i+1}v_{i+2}$ can be an induced one. Since there are no cycles of length $n-1$ in G, $v_{i-2}v_i$, $v_{i-1}v_{i+1}$, $v_iv_{i+2} \notin E(G)$, implying that $v_{i-2}v_{i+1}$ and $v_{i-1}v_{i+2}$ are edges in G. Now consider the path $P = v_{i+3}C^+v_{i-3}$. Clearly, $d_P(v_i) \geq n/2 - 2 = (|V(P)| + 1)/2$. If v_i is adjacent to two consecutive vertices of the path, say v_k and v_{k+1}, then the cycle $v_{i+1}C^+v_kv_iv_{k+1}C^+v_{i-2}v_{i+1}$ is a cycle of length $n-1$, a contradiction. This implies that $|V(P)|$ is odd and that the neighbourhood of v_i in P is $N_P(v_i) = \{v_{i+3}, v_{i+5}, ..., v_{i-5}, v_{i-3}\}$. Clearly, if $v_{i-1}v_{i+3} \in E(G)$, then there is a cycle of length $n-1$ in G, namely $v_{i+1}v_{i+2}v_{i-1}v_{i+3}C^+v_{i-2}v_{i+1}$. Thus $v_{i-1}v_{i+3} \notin E(G)$. But now $\{v_i; v_{i-1}, v_{i+1}, v_{i+3}\}$ induces a claw in G. Since neither v_{i-1} nor v_{i+1} is heavy, this contradicts G being a member of the family $\mathcal{F}(K_{1,3}, n)$. □

Claim 4.2. *If v_i is a heavy vertex in G, then $d_G(v_i) = n/2$.*

Proof. By Claim 4.1 we may assume that both v_i and v_{i+1} are heavy. If the degree of v_i is strictly greater than $n/2$, then $d_G(v_i) + d_G(v_{i+1}) \geq n + 1$ and so G is pancyclic by Lemma 2.3. This contradicts the assumption of G missing the $(n-1)$-cycle. □

Claim 4.3. *If v_i and v_{i+1} are heavy vertices in G, then none of the vertices v_{i-2}, v_{i-1}, v_{i+2} and v_{i+3} is heavy and the vertices v_{i+4} and v_{i+5} are both heavy.*
Furthermore, the path $v_{i-2}v_{i-1}v_iv_{i+1}v_{i+2}v_{i+3}$ is an induced one, and v_iv_{i-3}, v_iv_{i-4}, $v_{i+1}v_{i+4}$, $v_{i+1}v_{i+5}$ are edges in G.

Proof. Since G is missing the $(n-1)$-cycle, it is not bipartite and, by Claim 4.2, the degrees of both v_i and v_{i+1} are equal to $n/2$, it follows by Lemma 2.4 that $v_{i-2}v_{i-1}v_iv_{i+1}v_{i+2}v_{i+3}$ is an induced path P_6 in G and that v_i is adjacent to v_{i-3} and v_{i-4}, and v_{i+1} is adjacent to v_{i+4} and v_{i+5}. Thus, the last part of the claim holds. For a proof of the first part suppose that v_{i-1} is heavy. Then applying Lemma 2.4 to the pair $\{v_{i-1}, v_i\}$ leads to a contradiction with the adjacency structure it provides, since $v_iv_{i+3} \notin E(G)$. Similar contradiction arises if we suppose that v_{i+2} is heavy and apply Lemma 2.4 to the pair v_{i+1}, v_{i+2}. Thus neither v_{i-1} nor v_{i+2} is heavy. Now suppose that v_{i-2} is heavy. From the previous observation and from Claim 4.1 it follows that v_{i-3} is also heavy. Since $v_{i-2}v_{i+1} \notin E(G)$ this again leads to a contradiction with the structure described by Lemma 2.4, when applied to the pair $\{v_{i-2}, v_{i-3}\}$. Similar contradiction is obtained if one assumes that v_{i+3} is heavy. Thus the first part of the claim holds.

Now it will be shown that v_{i+4} is heavy. Suppose to the contrary that $d_G(v_{i+4}) < n/2$. Since the degree of v_{i+2} is also less than $n/2$, the path $v_{i+2}v_{i+3}v_{i+4}v_{i+5}$ can not be an induced one. This implies that $v_{i+2}v_{i+5} \in E(G)$. Similarly, to avoid induced claw

$\{v_{i+1};\, v_i,\, v_{i+2},\, v_{i+4}\}$, v_i is adjacent to v_{i+4}. But these two edges create in G a cycle of length $n-1$, namely $v_{i+2}v_{i+5}C^+v_iv_{i+4}v_{i+3}v_{i+2}$, a contradiction. Thus v_{i+4} is heavy. Since $d_G(v_{i+3}) < n/2$, the heaviness of v_{i+5} follows from Claim 4.1. □

Since there is a heavy vertex in G, we can assume without loss of generality that the vertices v_0 and v_1 are heavy, by Claim 4.1. It follows from Claim 4.3 that v_4 and v_5 are also heavy. Applying Claim 4.3 to the pair $\{v_4,\, v_5\}$ we obtain the heaviness of the vertices v_8 and v_9, and so on, i.e., every vertex v_j of G with $j \in \{4k,\, 4k+1\}$ for some non-negative integer k is heavy. Similarly, every $v_j \in V(G)$ with $j \in \{4k+2,\, 4k+3\}$ is not heavy. Thus the number of vertices of G is divisible by four. Let $n = 4r$, with $r > 2$. Then the set of heavy vertices of G is $\{v_0\, v_1,\, v_4,\, v_5,\, ...,\, v_{4r-4},\, v_{4r-3}\}$ and the remaining vertices are not heavy.

Claim 4.4. *Every heavy vertex of G is adjacent to exactly one non-heavy vertex.*

Proof. Suppose the contrary. Let v_i be a heavy vertex of G with at least two non-heavy neighbours. From Claims 4.1, 4.2 and 4.3 it follows that at least one of these neighbours, say v_k, satisfies $d_C(v_i,\, v_k) \geq 5$. Claims 4.1 and 4.3 imply that exactly one of the vertices v_{i-1} and v_{i+1} is also not heavy. Thus $\{v_i;\, v_{i-1},\, v_{i+1},\, v_k\}$ can not induce a claw, since $G \in \mathcal{F}(K_{1,3},\, n)$. Since there are no cycles of length $n-1$ in G, it follows that v_kv_{i-1} or v_kv_{i+1} is an edge in G.

Depending on which of the vertices v_{k-1} and v_{k+1} is heavy, either $v_{k-1}v_{k+2}$ or else $v_{k-2}v_{k+1}$ is an edge in G, by Claim 4.3. Denote this edge w_1w_2. This, together with the previous observations, implies that either $v_iC^+w_1w_2C^+v_{i-1}v_kv_i$ or $v_iC^-w_2w_1C^-v_{i+1}v_kv_i$ is a cycle in G. Since the length of this cycle is $n-1$, this contradicts the assumption of G missing the $(n-1)$-cycle. □

Claim 4.4 implies that, since there are $2r$ heavy vertices and $2r$ non-heavy vertices in G, in order for the heavy vertices to be indeed heavy, every two of them are adjacent. Thus the heavy vertices induce a clique in G and there is a perfect matching between this clique and the set of non-heavy vertices, since every heavy vertex has a non-heavy neighbour that lies next to it on the cycle C. Clearly, every non-heavy vertex v has a unique non-heavy neighbour u with $d_C(v,\, u) = 1$. To complete the proof it suffices to show that every non-heavy vertex is adjacent to exactly one non-heavy vertex.

Suppose this is not the case. Let v_k be a non-heavy vertex with v_{k+1} being also not heavy. Suppose v_k has a neighbour in a pair of non-heavy vertices $\{v_m,\, v_{m+1}\}$. From Claim 4.3 it follows that $d_C(v_k,\, v_m) \geq 7$. Since the heavy vertices of G induce a clique, either $v_kv_mv_{m-1}v_{m+2}C^+v_{k-1}v_{m-2}C^-v_k$ or $v_kv_{m+1}C^+v_{k-1}v_{m-1}C^-v_k$ is a cycle in G. The length of this cycle is $n-1$. This final contradiction completes the proof of Lemma 4.2. □

Lemma 4.3 (WW [51])**.** *Let G be a 2-connected graph of order $n \geq 3$ with at least two heavy vertices. If $G \in \mathcal{F}(\{K_{1,3},\, P_4\},\, n)$, then*
(i) if G is not bipartite, then G contains a triangle,
(ii) there is a cycle of length four in G.

Proof. For the proof of (i) assume that G is not bipartite. As the statement is easy to verify for $n \leq 4$, we further assume that $n \geq 5$. Let u be one of the heavy vertices in G. Clearly, if there is an edge in the subgraph of G induced by the neighbourhood $N_G(u)$ of u, then there is a triangle in G. Suppose that $G[N_G(u)]$ is edgeless. Since $G \in \mathcal{F}(K_{1,3}, n)$, it follows that at most one of the neighbours of u is not heavy. Observe that G is hamiltonian by Theorem 1.19. Let $C = v_1 ... v_n v_1$ be a hamiltonian cycle in G with $v_1 = u$. Since at least one of the vertices v_2 and v_n is heavy, Lemma 2.4 implies that there is a triangle in G.

Now it will be shown that (ii) holds. Let u and v be heavy vertices in G. Clearly, if u and v have at least two common neighbours, then G contains C_4. Thus suppose they have at most one common neighbour. Since both u and v are heavy, it follows that $uv \in E(G)$. If u and v have no common neighbours, then $V(G) = A \cup B \cup \{u, v\}$, where $N_G(u) = A \cup \{v\}$, $N_G(v) = B \cup \{u\}$ and $A \cap B = \emptyset$. Since G is 2-connected, there is an edge ab in G for some $a \in A$ and $b \in B$. This edge creates the cycle $uabvu$ of length four.

Assume that there is exactly one common neighbour of u and v in G, say w. Let $N_G(u) = A \cup \{v, w\}$ and $N_G(v) = B \cup \{u, w\}$, where $A \cap B = \emptyset$. Furthermore, assume that $N_G[w] \cap (A \cup B) = \emptyset$ and that there are no edges between the sets A and B, since otherwise there is a cycle of length four in G. From the 2-connectivity of G it follows that there is a path connecting A and B that is disjoint with the vertices u and v. Hence, there is a vertex in $V(G)$ that does not belong to $A \cup B \cup \{u, v, w\}$. This implies that

$$|A| + |B| + 3 < n.$$

On the other hand, since u and v are heavy, both A and B contain at least $n/2 - 2$ vertices. Thus

$$|A| + |B| + 4 \geq n.$$

Hence, $|A| + |B| + 4 = n$, $|A| = |B| = n/2 - 2$, and there is exactly one vertex, say x, in the set $V(G) \setminus (A \cup B \cup \{u, v, w\})$. In order to create a path between A and B with the set of vertices disjoint with both u and v, the vertex x is adjacent to some $a \in A$ and some $b \in B$. Hence, there is an induced path $uaxb$ in G. Since none of the vertices from $A \cup B$ is heavy, this contradicts G belonging to the family $\mathcal{F}(P_4, n)$. □

Now we are ready to prove Theorem 1.20.

Proof of Theorem 1.20: Let G be a graph satisfying the assumptions of the Theorem. Assume that G is not one of $K_{n/2, n/2}$, $K_{n/2, n/2} - e$ and F_{4r}. Lemmas 4.1 and 4.2 imply that G is neither bipartite nor missing the $(n-1)$-cycle. Furthermore, there is a hamiltonian cycle in G, by Theorem 1.19.

Toward a contradiction, suppose that G is not pancyclic. Then it follows from Theorem 1.11 that G is not $\{K_{1,3}, P_4\}$-free and so there are at least two heavy vertices in G. The following claim gathers the pieces of information regarding cycles in G that we have obtained so far.

Claim 4.5. *G contains cycles of lengths three, four, $n-1$ and n.*

Proof. The existence of the long cycles is clear. The fact that there are cycles C_3 and C_4 in G follows from Lemma 4.3. □

By Claim 4.5, if $n \leq 6$, then G is pancyclic. So we assume that $n \geq 7$.

Claim 4.6. *If $x, y \in V(G)$ are heavy in G, then for every hamiltonian cycle C in G holds $d_C(x, y) \geq 2$. Furthermore, if $d_C(x, y) = 2$, then $d_G(x) = d_G(y) = n/2$ and $xy \in E(G)$.*

Proof. Clearly, if $d_C(x, y) = 1$, then G is pancyclic by Lemma 2.4. If $d_C(x, y) = 2$ and the degree of at least one of x and y is strictly greater than $n/2$, then G is pancyclic by Lemma 2.5. Finally, if $d_C(x, y) = 2$ and x is not adjacent to y, pancyclicity of G follows from Claim 4.5 and Lemma 2.6. □

Let u be a vertex in G with $d_G(u) \geq n/2$.

Case 1: $G - u$ is not 2-connected.

Under the assumptions of this case there is a vertex in G, say v, such that $G - \{u, v\}$ is not connected. Since G is hamiltonian, we can set $C = uy_1...y_{h_2}vx_{h_1}...x_1u$ to be a hamiltonian cycle with $H_1 = \{x_1, ..., x_{h_1}\}$ and $H_2 = \{y_1, ..., y_{h_2}\}$ being the components of $G - \{u, v\}$. The following simple observation is crucial for the further reasoning.

Claim 4.7. *There are no heavy vertices in at least one of the sets H_1 and H_2.*

Proof. Suppose this is not the case. Then $h_1 = h_2 = (n-2)/2$ and there are vertices $x \in H_1$, $y \in H_2$ such that $N_G(x) = H_1 \cup \{u, v\}$ and $N_G(y) = H_2 \cup \{u, v\}$. Thus $uyvxu$ is a cycle of length four in G. To this cycle all vertices from H_2 can be appended, one-by-one, creating cycles uy_1yvxu, uy_1y_2yvxu, ..., $uC^+yy_{h_2}vxu$, $uC^+yy_{h_2-1}y_{h_2}vxu$, ..., uC^+vxu. The vertices from H_1 can be appended to the longest of these cycles in a similar manner. In this way we obtain $[4, n]$-cycles in G. Since G contains a triangle, by Claim 4.5, it is pancyclic. A contradiction. □

It follows from Claim 4.7 that for the rest of the proof of this case we may assume a lack of heavy vertices in H_1. We also assume that y_1 is not heavy, since the opposite yields a contradiction with Claim 4.6.

The next three claims describe the neighbourhood $N_G(u)$ of the vertex u.

Claim 4.8. $N_{H_2}[u] \subset N_G[y_1]$.

Proof. Otherwise u is adjacent to some vertex $y \in H_2 \setminus N_G[y_1]$. Then $\{u; y, y_1, x_1\}$ induces a claw in G. Since neither x_1 nor y_1 is heavy, this contradicts G being a member of the family $\mathcal{F}(K_{1,3}, n)$. □

Claim 4.9. $N_{H_1}(u) = H_1$ *and* $N_{H_1}[u]$ *induces a clique.*

Proof. Since the statement is clearly true for $h_1 = 1$, assume that there are at least two vertices in H_1. Suppose that there is a vertex $x_i \in H_1$ such that $ux_i \notin E(G)$. Choose minimal i with this property. Then the path $y_1ux_{i-1}x_i$ is induced in G. Since there are no heavy vertices in H_1 and y_1 is not heavy, this is a contradiction with G belonging to the family $\mathcal{F}(P_4, n)$.

Now suppose that there are two nonadjacent neighbours of u in H_1, say x and x'. Then $\{u; x, x', y_1\}$ induces a claw, with none of its endvertices being heavy. Since $G \in \mathcal{F}(K_{1,3}, n)$, this is a contradiction. $\square$

Claim 4.10. *$N_{H_2}(u) \neq H_2$.*

Proof. Suppose the contrary. Then $uy_{h_2}vx_{h_1}u$ is a cycle in G, by Claim 4.9. To this cycle we can append the vertices from H_1, one-by-one, also by Claim 4.9. To the longest of the cycles obtained the vertices from H_2 can be appended in a similar way. With this procedure we obtain $[4, n]$-cycles in G. The pancyclicity of G follows from Claim 4.5. $\square$

It follows from Claim 4.10 that there is a vertex y_k in $N_{H_2}(u)$ such that $y_{k+1} \in H_2$ and u is not adjacent to y_{k+1}. Choose minimal k satisfying these conditions.

Claim 4.11. *y_k is heavy. In consequence, $k \geq 2$, both y_{k-1} and y_{k+1} are not heavy, and $y_{k-1}y_{k+1} \notin E(G)$.*

Proof. Clearly, the path $x_1uy_ky_{k+1}$ is an induced one. Since $G \in \mathcal{F}(P_4, n)$ and x_1 is not heavy, it follows that y_k is heavy. Now Claim 4.6 implies that $k \geq 2$ and that neither y_{k-1} nor y_{k+1} is heavy. The fact that y_{k-1} is not adjacent to y_{k+1} follows from Lemma 2.1. $\square$

Claim 4.12. *There are $[n - h_1, n]$-cycles in G.*

Proof. Claim 4.11 implies that u is adjacent to y_2. Thus $C' = uy_2C^+x_{h_1}u$ is a cycle of length $n - h_1$, by Claim 4.9. Since u is adjacent to every vertex of H_1, all these vertices can be appended to C', one-by-one, creating cycles of demanded lengths. $\square$

Claim 4.13. *$uv \in E(G)$.*

Proof. Suppose the contrary. Then $y_1v \in E(G)$ to avoid induced path $y_1ux_{h_1}v$ with neither y_1 nor x_{h_1} being heavy. Now it follows from Claims 4.8 and 4.9 that $d_G(y_1) \geq n/2 - h_1 + 1$. Set $G' = G - \{x_1, ..., x_{h_1-1}\}$ if $h_1 > 1$ or $G' = G$ otherwise. Since $y_1y_k \in E(G)$, by Claim 4.8, G' is hamiltonian, with $C' = uy_{k-1}C^-y_1y_kC^+x_{h_1}u$ being its hamiltonian cycle. Note that $d_{G'}(y_1) + d_{G'}(y_k) \geq n/2 - h_1 + 1 + n/2 = |G'|$, by Claim 4.11, and that uy_1y_2u is a triangle in G'. Thus it follows from Lemma 2.4 that G' is either pancyclic or else missing only $(n - h_1)$-cycle. In either case Claim 4.12 implies pancyclicity of G. $\square$

The next claim provides a full description of the neighbourhood of the vertex y_1.

Claim 4.14. *$N_G[y_1] = N_{H_2}[u]$.*

Proof. Suppose that the claim is not true. Claim 4.8 implies that y_1 is adjacent to v or to some vertex $y \in h_2$ which is a non-neighbour of u. It follows from Claims 4.8, 4.9 and 4.13 that $d_G(y_1) \geq n/2 - h_1 - 1 + 1 = n/2 - h_1$. By Claim 4.13 the cycle $uy_{k-1}C^-y_1y_kC^+vu$ is a hamiltonian cycle in $G' = G - H_1$. Since $d_{G'}(y_1) + d_{G'}(y_k) \geq |G'|$ and uy_1y_2u is a triangle in G', it follows from Lemma 2.4 that G' is either pancyclic or else missing only $(n - h_1 - 1)$-cycle. By Claim 4.12, the same is true for G. Since uy_2C^+vu is a cycle of length $n - h_1 - 1$, G is pancyclic. □

Claim 4.15. *y_{h_2} is adjacent neither to u nor to y_1.*

Proof. Suppose this is not the case. Then, by Claim 4.14, y_{h_2} is adjacent to both u and y_1. If $vy_k \notin E(G)$, then set $G' = G - (H_1 \cup \{v\})$. Note that the cycle $uy_{k-1}C^-y_1y_kC^+y_{h_2}u$ is a hamiltonian cycle in G' and $uy_2C^+y_{h_2}u$ is a cycle of length $|G'| - 1$. Since $d_{G'}(y_1) + d_{G'}(y_k) \geq |G'|$, Lemma 2.4 implies that G' is pancyclic. Together with Claim 4.12 this implies pancyclicity of G.

Now assume $vy_k \in E(G)$. If $vy_{k-1} \in E(G)$, then consider $G' = G - H_1$. Again, G' is a graph with both $|G'|$- and $(|G'| - 1)$-cycles, namely, $vy_{k-1}C^-uy_kC^+v$ and uy_2C^+vu. Clearly, $d_{G'}(u) + d_{G'}(y_k) \geq |G'|$, by Claim 4.11, and G' is not bipartite. Thus G' is pancyclic, by Lemma 2.4, and so G is pancyclic, by Claim 4.12.

Hence, v is adjacent to y_k and not adjacent to y_{k-1}. Now to avoid $\{y_k;\, y_{k-1},\, v,\, y_{k+1}\}$ inducing a claw with neither y_{k-1} nor y_{k+1} being heavy, v is adjacent to y_{k+1}. But then $y_{k+1}vC^+uy_kC^-y_1y_{h_2}C^-y_{k+1}$ is a hamiltonian cycle in G with both u and y_k being heavy. This contradicts Claim 4.6. □

Observe that, by Claims 4.13, 4.14 and 4.15, the path $y_1uvy_{h_2}$ is an induced one. Since y_1 is not heavy, it follows that v is heavy. In consequence, y_{h_2} is not heavy, by Claim 4.6.

Claim 4.16. *y_{h_2} is adjacent to both y_k and y_{k+1}.*

Proof. We first observe that $vy_{h_2-1} \in E(G)$. Clearly, otherwise the path $x_{h_1}vy_{h_2}y_{h_2-1}$ is an induced one. Since neither x_{h_1} nor y_{h_2} is heavy, this contradicts G belonging to the family $\mathcal{F}(P_4,\, n)$.

Now suppose that y_{h_2} is not adjacent to y_k. Set $G' = G - (H_1 \cup y_{h_2})$. It follows from Claims 4.11, 4.13, 4.14 and 4.15 that $d_{G'}(y_1) + d_{G'}(y_k) \geq |G'|$. Since $y_1y_kC^+y_{h_2-1}vuy_{k-1}C^-y_1$ is a hamiltonian cycle and $uy_2C^+y_{h_2-1}vu$ is a cycle of length $|G'| - 1$ in G', Lemma 2.4 implies pancyclicity of G'. Thus there are $[3,\, n - h_1 - 1]$-cycles in G and so G is pancyclic, by Claim 4.12.

Hence, $y_{h_2}y_k \in E(G)$. Suppose $y_{h_2}y_{k+1} \notin E(G)$. It follows from Claims 4.14 and 4.15 and the choice of k that $\{y_k;\, y_1,\, y_{h_2},\, y_{k+1}\}$ induces a claw. Since none of the endvertices of this claw is heavy, this is a contradiction. Thus y_{h_2} is adjacent to y_{k+1}. □

Claim 4.17. *v is adjacent to every vertex from the set $\{y_k,\, y_{k+1},\, ...,\, y_{h_2}\}$.*

Proof. Suppose that the above statement is not true. Then there is a vertex $y_m \in N_{H_2}(v)$ such that $y_{m-1} \in \{y_k,\, y_{k+1},\, ...,\, y_{h_2-1}\}$ and $vy_{m-1} \notin E(G)$. Choose maximal m satisfying these conditions. Note that, since v is heavy and $G - v$ is not 2-connected, we could change u with v in the beginning of the proof of this case and repeat the reasoning conducted so far, obtaining in particular that $N_{H_1}(v) = H_1$, and $N_{H_2}(v) \neq H_2$. Then y_m would be an equivalent of y_k for u, and thus we could show that y_m is heavy, and so on. Finally, similarly to Claim 4.16, i.e., the existence of the edge $y_{h_2}y_{k+1}$, by symmetry we would obtain the existence of the edge y_1y_{m-1}. But then the cycle $uy_kC^-y_1y_{m-1}C^-y_{k+1}y_{h_2}C^-y_mvC^+u$ is a hamiltonian cycle in G with $d_G(u) + d_G(y_k) \geq n$, a contradiction with Claim 4.6. □

Now it follows from Claim 4.17 that uy_kvu is a triangle in G. Since $\{y_1, ..., y_{k-1}\} \subset N_G(u)$ and $\{y_{k+1}, ..., y_{h_2}\} \subset N_G(v)$, we can append the vertices from H_2 to this triangle, one-by-one, obtaining cycles of all lengths from three up to $h_2 + 2 = n - h_1$. Since there are also $[n - h_1, n]$-cycles in G, by Claim 4.12, this implies that G is pancyclic. This final contradiction completes the proof of this case.

Case 2: $G - u$ is 2-connected.

Set $G' = G - u$. Note that G' is not hamiltonian, by Lemma 2.1, and that for every heavy vertex v of G other than u we have $d_{G'}(v) \geq n/2 - 1 = (n - 2)/2$. Thus $G' \in \mathcal{F}(\{K_{1,3}, P_4\}, n - 2)$. It follows from Theorem 1.19 that there is a cycle of length $n - 2$ in G', say $C' = w_1w_2...w_{n-2}w_1$. In the following any arithmetic involving the subscripts of the vertices of C' is modulo $n - 2$.

Let x be the vertex of G' such that $x \notin V(C')$. Lemma 2.1 implies that $d_{G'}(x) \leq (n-2)/2$. Next we will show that this inequality is in fact strict.

Claim 4.18. *$d_{G'}(x) < (n - 2)/2$.*

Proof. Suppose that the above statement is not true, i.e., that $d_{G'}(x) = (n - 2)/2$. Since G' is not hamiltonian, we can assume $N_{C'}(x) = \{w_1, w_3, ..., w_{n-3}\}$. It is not difficult to see, that if u is joined by an edge with two consecutive vertices of C', then G is pancyclic. Thus

$$n/2 \leq d_G(u) = d_{C'}(u) + e(u, x) \leq (n - 2)/2 + 1 = n/2,$$

implying that $ux \in E(G)$ and u is joined to either each vertex of the set $\{w_1, w_3, ..., w_{n-3}\}$ or else to each vertex of the set $\{w_2, w_4, ..., w_{n-2}\}$. If the first case occurs, then G is clearly pancyclic. Thus assume the latter is true. Since G is not bipartite, there is a chord in C' joining two vertices whose indices have the same parity. One can easily check that G is pancyclic. □

Claim 4.19. *$ux \in E(G)$ and $d_G(u) = n/2$.*

Proof. If at least one of the above conditions is not satisfied, then $d_{C'}(u) \geq (n - 1)/2$, implying pancyclicity of $G - x$, by Lemma 2.1, and, in consequence, pancyclicity of G. □

Fix k for which there are no k-cycles in G. It follows from Claim 4.5 and the existence of C' that $k \in \{5, 6, ..., n - 3\}$. Furthermore, for every i from the set $\{1, ..., n - 2\}$ we have $e(u, w_i) + e(u, w_{i+k-2}) \leq 1$, since otherwise $uw_iC'^{+}w_{i+k-2}u$ is a cycle C_k. This implies, together with Claim 4.19, that

$$n-2 \leq 2d_{C'}(u) = \sum_{i=1}^{n-2}[e(u, w_i) + e(u, w_{i+k-2})] \leq n-2.$$

Thus $d_{C'}(u) = (n-2)/2$ and the following holds:

$$\forall i \in \{1, ..., n-2\}\colon e(u, w_i) + e(u, w_{i+k-2}) = 1 \tag{1}$$

We also note that in order to avoid the cycle $xw_iC'^{+}w_{i+k-3}ux$ of length k, for every i with $1 \leq i \leq n-2$ the following inequality holds:

$$e(x, w_i) + e(u, w_{i+k-3}) \leq 1 \tag{2}$$

Now we examine relations between the vertices u and x and the vertices of the cycle C'.

Claim 4.20. *Let l be an integer satisfying $1 \leq l \leq k-4$. If w_i is a neighbour of x in $V(C')$, then*
(i) $xw_{i-l} \notin E(G)$,
(ii) $uw_{i-l} \in E(G)$,
(iii) w_{i-l} is not heavy in G,
(iv) $w_{i-l}w_{i+1} \in E(G)$.

Proof. The proof is by induction on l. Clearly, xw_{i-1}, $xw_{i+1} \notin E(G)$ to avoid hamiltonian cycle in G'. Since x is adjacent to w_i, it follows from (2) that $uw_{i+k-3} \notin E(G)$. Thus, by (1), u is adjacent to w_{i-1}. Note that $uxw_iC'^{+}w_{i-1}u$ is a hamiltonian cycle in G. Since u is heavy, Claim 5.3 implies that w_{i-1} is not heavy. To prove (iv) observe that if $w_{i-1}w_{i+1}$ is not an edge in G, then $\{w_i; w_{i-1}, x, w_{i+1}\}$ induces a claw. Since neither w_{i-1} nor x, by Claim 4.18, is heavy, this contradicts G being a member of the family $\mathcal{F}(K_{1,3}, n)$.

Assume that the Claim holds for the values from the set $\{1, 2, ..., l\}$ with l satisfying $l < k-4$. We will show that this implies the validity of the claim for $l+1$.

Suppose $xw_{i-l-1} \in E(G)$. Then, by the condition (iv) for l, there is a hamiltonian cycle in G', namely $xw_{i-l-1}C'^{-}w_{i+1}w_{i-l}C'^{+}w_ix$. This contradiction proves (i).

By the conditions (i) and (ii) the vertex w_{i-l} is adjacent to u and not adjacent to x. Thus $uw_{i-l-1} \in E(G)$ to avoid induced path $xuw_{i-l}w_{i-l-1}$ with neither x nor w_{i-l} being heavy. This proves (ii). Now, since $uw_{i-l-1} \in E(G)$ and, by (iv), w_{i-l} is adjacent to w_{i+1}, the cycle $uw_{i-l-1}C'^{-}w_{i+1}w_{i-l}C'^{+}w_ixu$ is a hamiltonian cycle in G. Since u is heavy, Claim 4.6 implies that w_{i-l-1} is not heavy.

For the proof of (iv) suppose that w_{i-l-1} is not adjacent to w_{i+1}. Note that $uw_{i+1} \in E(G)$ to avoid induced path $xuw_{i-1}w_{i+1}$ with neither x nor w_{i-1} being heavy in G. But this implies that $\{u; x, w_{i-l-1}, w_{i+1}\}$ induces a claw in G. Since none of the vertices x and w_{i-l-1} is heavy, this contradicts G belonging to the family $\mathcal{F}(K_{1,3}, n)$. By mathematical induction the Claim is true.

□

Since G is 2-connected, there is a vertex $w_i \in V(C')$ adjacent to x. From Claim 4.20 it follows that $uxw_iw_{i+1}w_{i-1}C'^{-}w_{i-k+4}u$ is a cycle in G. Since the length of this cycle is k, this contradicts the choice of k. This final contradiction completes the proof. $\square$

5 Proof of Theorem 1.27

As usual, we begin with repeating the theorem under consideration.

Theorem 1.27 (WW) *Let G be a 2-connected graph with n vertices. If $G \in \mathcal{F}(\{K_{1,3}, P_7\}, n+1)$ and*

1. *$n \geq 14$ and $G \in \mathcal{F}(D, n+1)$, or*
2. *$G \in \mathcal{F}(H, n+1)$ and there is a super-heavy vertex in G,*

then G is pancyclic.

Proof of Theorem 1.27: The theorem will be proved by contradiction. Suppose that a graph G with n vertices satisfies the assumptions of the theorem but is not pancyclic.

Claim 5.1. *There is a super-heavy vertex u in G and a vertex $v \in V(G) \setminus \{u\}$ such that $G - \{u, v\}$ is not connected.*

Proof. Suppose that G satisfies the first of the assumptions of the theorem, i.e., that $G \in \mathcal{F}(D, n+1)$ and $n \geq 14$. Then it follows from Theorem 1.15 that G is not $\{K_{1,3}, P_7, D\}$-free, and so there is a super-heavy vertex in G, say u. Note that $G-u \in \mathcal{F}(\{K_{1,3}, P_7, D\}, n-1)$. If $G - u$ is 2-connected, then it is hamiltonian by Theorem 1.23 and so G is pancyclic by Lemma 2.1. Hence, there is a vertex $v \in V(G) \setminus \{u\}$ such that $G - \{u, v\}$ is not connected.

Now suppose that G satisfies the second one of the assumptions. Let $u \in V(G)$ be a super-heavy vertex in G. As in the previous case, we observe that $G - u$ belongs to the family $\mathcal{F}(\{K_{1,3}, P_7, H\}, n-1)$ and so it is not 2-connected, by Theorem 1.23 and Lemma 2.1. The claim follows. $\square$

Note that G is hamiltonian, by Theorem 1.23 or 1.24. Let C be a hamiltonian cycle in G. By Claim 5.1 we can choose a super-heavy vertex $u \in V(G)$ and a vertex $v \in V(G)\setminus\{u\}$ and set $C = uy_1...y_{h_2}vx_{h_1}...x_1u$, where $H_1 = \{x_1, ..., x_{h_1}\}$ and $H_2 = \{y_1, ..., y_{h_2}\}$ are components of $G - \{u, v\}$. Assume, without loss of generality, that $h_1 \leq h_2$.

In the next seven claims we present some pieces of information on the structure of G, that will be of use throughout the rest of the proof.

Claim 5.2. *There are no super-heavy vertices in H_1.*

Proof. Clearly, the neighbourhood of a vertex $x \in H_1$ is a subset of the set $(H_1 - x) \cup \{u, v\}$. Since $h_1 \leq (n-2)/2$, it follows that $d_G(x) \leq n/2$. $\square$

Claim 5.3. *There are no super-heavy pairs of vertices with distance one or two along a hamiltonian cycle in G.*

Proof. Otherwise G is pancyclic by Lemma 2.3 or Lemma 2.5, a contradiction. $\square$

Claim 5.4. $N_{H_2}[u] \subseteq N_G[y_1]$.

Proof. Suppose this is not true. Then there is a vertex $y \in N_{H_2}(u) \setminus N_G[y_1]$. But now $\{u;\, x_1,\, y_1,\, y\}$ induces a claw. Since G belongs to the family $\mathcal{F}(K_{1,3},\, n+1)$ and x_1 is not super-heavy, it follows that y_1 is super-heavy. Since $d_C(u,\, y_1) = 1$, this contradicts Claim 5.3. □

Claim 5.5. *If $y_i y_{i+2} \notin E(G)$ for some vertices $y_i,\, y_{i+2} \in H_2$, then at least one of them is not adjacent to u.*

Proof. Otherwise $\{u;\, x_1,\, y_i,\, y_{i+2}\}$ induces a claw. Since $G \in \mathcal{F}(K_{1,3},\, n+1)$ and x_1 is not super-heavy by Claim 5.2, both y_i and y_{i+2} are super-heavy. This contradicts Claim 5.3. □

Claim 5.6. *$N_{H_1}[u]$ induces a clique in G.*

Proof. Since the statement is obvious for $h_1 = 1$ and $h_1 = 2$, assume $h_1 \geq 3$. Suppose the claim is not true, i.e. that there exist vertices $x_a,\, x_b \in N_{H_1}(u)$ such that $x_a x_b \notin E(G)$. Then $\{u;\, x_a,\, x_b,\, y_1\}$ induces a claw. Since neither x_a nor x_b is super-heavy, by Claim 5.2, this contradicts G being a member of the family $\mathcal{F}(K_{1,3},\, n+1)$. □

Claim 5.7. *$N_{H_2}(u) \neq H_2$.*

Proof. Otherwise there are both $[3,\, h_2+1]-$ and $[n-h_2+1,\, n]$-cycles in G. If $h_2 > (n-2)/2$, this implies that G is pancyclic, a contradiction. Since $h_2 \geq (n-2)/2$, it follows that $h_2 = (n-2)/2 = h_1$ and G is missing only the (h_2+2)-cycle. Now, if u is adjacent to some vertex $x_i \in H_1$ other than x_1, then $u y_{n-i-h_2} C^+ x_i u$ is a cycle of length h_2+2, a contradiction. Thus $d_{H_1}(u) = 1$. Since u is super-heavy, this implies that uv is an edge in G. But then uvC^+u is a cycle of length h_2+2.

□

We distinguish four cases, depending on the number of vertices in H_1 and the number of neighbours of u in H_1.

We begin with a case when $h_1 = 1$. It is showed that under this assumption one can find an in induced path P_7 with five of its vertices being consecutive vertices of the cycle C. This fact leads to a contradiction with Claim 5.3.

In Subcase 2.1 it is assumed that $h_1 \geq 2$ and $d_{H_1}(u) = 1$. Using Lemma 2.4 and Claim 5.4 we prove that this implies that there are no one-chords in C, and, in consequence, that $h_1 = 3$. Then we obtain an induced claw which does not satisfy the Fan's condition.

The most complex parts of the proof are Subcases 2.2.1 and 2.2.2, where $h_1 \geq 2$ and $d_{H_1}(u) \geq 2$. The idea of the proof is to find a short cycle in G that can be extended by appending to it all the vertices of G, one-by-one (and thus creating all cycles of greater lengths). Firstly, the neighbourhood of u in G is examined. Then we prove that the nonneighbours of u can be used for extending the desired cycle. After the existence of short cycles in G is proved, we extend the longest of these short cycles using the observations made before and arrive at the conclusion of pancyclicity of G.

Case 1: $h_1 = 1$.

Claim 5.8. $uv \in E(G)$.

Proof. Suppose the contrary. Then, by Claim 5.4, we have $d_G(y_1) \geq (n-1)/2$ and so $d_G(u) + d_G(y_1) \geq n$. Lemma 2.4 implies that G is either bipartite or missing $(n-1)$-cycles. Suppose that there are indeed no cycles of length $n-1$ in G. This implies in particular that u is not adjacent to y_2. Since y_2 is a neighbour of y_1, it follows from Claim 5.4 that $d_G(y_1) \geq (n+1)/2$, a contradiction with Claim 5.3. Hence, there is a cycle of length $n-1$ in G. Since C is a cycle of length n, G is not bipartite. A contradiction. □

Recall that $N_{H_2}[u] \subseteq N_G[y_1]$ by Claim 5.4, implying $d_G(y_1) \geq (n+1)/2 - 2$ (since u is super-heavy and both x_1 and v are its neighbours) and $d_G(u) + d_G(y_1) \geq n-1$. We will refer to the latter implicitly in the following.

Claim 5.9. $N_{H_2}[u] = N_G[y_1]$.

Proof. Suppose that the claim is not true. Then, by Claim 5.4, either there is a vertex $y \in H_2$ adjacent to y_1 and not adjacent to u or else $vy_1 \in E(G)$. In either case it follows that $d_G(y_1) \geq (n+1)/2 - 1$ and so $d_G(u) + d_G(y_1) \geq n$. Since G is hamiltonian and uC^+vu is a cycle of length $n-1$, G is neither bipartite nor missing $(n-1)$-cycles. Lemma 2.4 implies that G is pancyclic, a contradiction. □

Claim 5.10. *There are* $[n-2,\, n]$*-cycles in* G.

Proof. Obviously, G is hamiltonian and vuC^+v is an $(n-1)$-cycle. Claim 5.9 implies that $uy_2 \in E(G)$ and so uy_2C^+vu is a cycle of length $n-2$. □

By Claim 5.7 we can choose a vertex $y_k \in N_{H_2}(u)$ such that $y_{k+1} \in H_2$ and $uy_{k+1} \notin E(G)$. Choose the minimal possible k for which this property holds. Observe that Claim 5.9 implies $k \geq 2$.

Claim 5.11. $h_2 \geq k+4$.

Proof. Suppose the contrary. Then $h_2 \in \{k+1,\, k+2,\, k+3\}$ and uy_kC^+vu is a cycle in G of length at most six. To this cycle all vertices from G can be appended, one-by-one, creating cycles of all lengths from 6 up to n, namely the cycles $uy_kC^+vx_1u$, $uy_{k-1}y_kC^+vx_1u$, ..., uy_2C^+u and C. Clearly, if $k \geq 5$, then pancyclicity of G follows from the choice of k. Assume $k \leq 4$. Since $n \leq k+6$, it follows that $n \leq 10$. This implies that G does not satisfy the first of the assumptions of the theorem, and so $G \in \mathcal{F}(H,\, n+1)$. Note that it follows from Claim 5.3 that y_1 is not super-heavy. Since x_1 is also not super-heavy, by Claim 5.2, the set $\{u;\, x_1,\, v;\, y_1,\, y_2\}$ can not induce H. This implies, together with Claim 5.9, that v is adjacent to y_2. But now it is easy to see that there are also $[3,\, 5]$-cycles in G, namely the cycles vx_1uv, vuy_1y_2v and vC^+y_2v.

□

Claim 5.12. $uy_{k+2} \notin E(G)$.

Proof. Suppose that the statement is not true. Then $uy_{k+2} \in E(G)$, implying, by Claim 5.5, that $y_ky_{k+2} \in E(G)$. Consider $G' = G - y_{k+1}$, a graph with the cycle $C' = uy_1C^+y_ky_{k+2}C^+u$ being its hamiltonian cycle. Since $uy_{k+1} \notin E(G)$ it follows from Claim 5.9 that $y_1y_{k+1} \notin E(G)$ and so

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) + d_G(y_1) \geq n - 1 = |G'|.$$

This implies, together with the fact that uC'^+vu is an $(|G'| - 1)$-cycle in G', that G' is pancyclic, by Lemma 2.4. But then G is also pancyclic, a contradiction. □

Claim 5.13. *y_ky_{k+2}, y_ky_{k+3}, $y_{k+1}y_{k+3} \notin E(G)$.*

Proof. This is indeed true, since if any of these edges exists, say y_ay_{a+i}, Lemma 2.9 for u, y_1, $X = \{y_{a+1}, y_{a+i-1}\}$ and a hamiltonian cycle $y_ay_{a+i}C^+y_a$ in $G - X$ implies pancyclicity of G. □

Claim 5.14. *$uy_{k+3} \notin E(G)$.*

Proof. Suppose that the statement is not true. Then it follows from Claim 5.13 that $\{u; x_1, y_k, y_{k+3}\}$ induces a claw. Since $G \in \mathcal{F}(K_{1,3}, n + 1)$ and x_1 is not super-heavy, by Claim 5.2, both y_k and y_{k+3} are super-heavy. But then G is pancyclic by Lemma 2.8 and Claim 5.13, a contradiction. □

Claim 5.15. *y_ky_{k+4}, $y_{k+1}y_{k+4}$, $y_{k+2}y_{k+4} \notin E(G)$.*

Proof. See proof of Claim 5.13 (which can now be applied here due to the Claim 5.14). □

Claim 5.16. *$uy_{k+4} \notin E(G)$.*

Proof. Suppose that the claim is not true. Then it follows from Claim 5.9 that $y_1y_{k+4} \in E(G)$ and from Claim 5.15 that $\{u; x_1, y_k, y_{k+4}\}$ induces a claw. Since $G \in \mathcal{F}(K_{1,3}, n + 1)$ and x_1 is not super-heavy by Claim 5.2, y_k is super-heavy.
Consider $G' = G - \{y_{k+1}, y_{k+2}, y_{k+3}\}$ with a hamiltonian cycle $y_1C^+y_kuC^-y_{k+4}y_1$. By Claims 5.12, 5.13, 5.14 and 5.15 and the fact that y_k is super-heavy we have

$$d_{G'}(u) + d_{G'}(y_k) = d_G(u) + d_G(y_k) - 1 \geq |G'| + 1.$$

Hence, G' is pancyclic by Lemma 2.3 and so there are $[3, n - 2]$-cycles in G. Together with Claim 5.10 this implies pancyclicity of G, a contradiction. □

Claims 5.12, 5.13, 5.14, 5.15 and 5.16 imply that the path $x_1uy_ky_{k+1}y_{k+2}y_{k+3}y_{k+4}$ is an induced one. Since G belongs to the family $\mathcal{F}(P_7, n + 1)$, it follows that at least two of the vertices y_{k+1}, y_{k+2}, y_{k+3} and y_{k+4} are super-heavy. Claim 5.3 implies that from these four vertices only y_{k+1} and y_{k+4} are super-heavy. But now the pancyclicity of G follows from Lemma 2.8 and Claim 5.10. This contradiction completes the proof of this case.

Case 2: $h_1 \geq 2$.

Subcase 2.1: $d_{H_1}(u) = 1$.

In this subcase the only neighbour of u in H_1 is x_1. As in the Case 1, Claim 5.4 implies that $d_G(y_1) \geq (n+1)/2-2$ and so $d_G(u)+d_G(y_1) \geq n-1$. Again, this fact will be implicitly referred to in the following.

Claim 5.17. $uv \in E(G)$.

Proof. Suppose the contrary. Then, by Claim 5.4, we have $d_G(y_1) \geq (n-1)/2$ and so $d_G(u) + d_G(y_1) \geq n$. Lemma 2.4 implies that G is either bipartite or missing $(n-1)$-cycles. Suppose that there are indeed no cycles of length $n-1$ in G. This implies in particular that u is not adjacent to y_2. Since y_2 is a neighbour of y_1, it follows from Claim 5.4 that $d_G(y_1) \geq (n+1)/2$, a contradiction with Claim 5.3. Hence, there is a cycle of length $n-1$ in G. Since C is a cycle of length n, G is not bipartite. A contradiction. □

Claim 5.18. *Suppose $x_i x_{i+2} \in E(G)$ for some x_i, $x_{i+2} \in H_1$. Then the only possible one-chords in C other than $x_i x_{i+2}$ are $x_{i-1}x_{i+1}$ and $x_{i+1}x_{i+3}$.*

Proof. Suppose that the claim is not true. Then there is a one-chord in C other than $x_{i-1}x_{i+1}$ and $x_{i+1}x_{i+3}$. Consider $G' = G - x_{i+1}$. Clearly, $C' = uy_1C^+x_ix_{i+2}C^+u$ is a hamiltonian cycle in G' with

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) + d_G(y_1) \geq |G'|.$$

Since the one-chord of C is also a one-chord in C', there is an $(|G'|-1)$-cycle in G'. Thus G' is not bipartite. Hence, G' is pancyclic by Lemma 2.4, implying pancyclicity of G: a contradiction. □

Claim 5.19. *Suppose $x_i x_{i+3} \in E(G)$ for some x_i, $x_{i+3} \in H_1$. Then there are no one-chords in C.*

Proof. Otherwise there is a one-chord in C. Let $G' = G - \{x_{i+1}, x_{i+2}\}$. Clearly, the cycle $uy_1C^+x_ix_{i+3}C^+u$ is a hamiltonian cycle in G'. Since

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) + d_G(y_1) \geq |G'| + 1,$$

Lemma 2.3 implies, that G' is pancyclic. Thus there are $[3, n-2]$-cycles in G. Since the one-chord in C creates a cycle of length $n-1$ and G is hamiltonian, G is pancyclic. A contradiction. □

Claim 5.20. *If there is a one-chord in $C[u, v]$, then there are no one-chords and no two-chords in $C[x_{h_1}, x_1]$.*

Proof. This Claim is a corollary of Claim 5.18 and Claim 5.19. □

Claim 5.21. *Suppose there is a one-chord in $C[u, v]$. Then $h_1 \leq 3$.*

Proof. Suppose the statement is not true. Then there is a one-chord in $C[u, v]$ and $h_1 \geq 4$. Recall that $y_k \in N_{H_2}(u)$ is such a vertex that $y_{k+1} \in H_2$ and $uy_{k+1} \notin E(G)$. Since $N_{H_1}(u) = \{x_1\}$, Claim 5.20 implies, that $\{x_4, x_3, x_2, x_1, u, y_k, y_{k+1}\}$ induces a P_7. Since neither x_4 nor x_2 are super-heavy, by Claim 5.2, this contradicts G belonging to the family $\mathcal{F}(P_7, n+1)$. □

Claim 5.22. *There are no one-chords in* $C[u, v]$.

Proof. Suppose that the claim is not true. Then there is a one-chord in $C[u, v]$ and so $h_1 \leq 3$, by Claim 5.21.

First assume $h_1 = 2$. Consider $G' = G - \{x_1, x_2\}$. By Claim 5.17 $C' = uy_1C^+vu$ is a hamiltonian cycle in G'. Since the one-chord in $C[u, v]$ is also a one-chord in C', there is a cycle of length $|G'| - 1$ in G'. Furthermore, we have

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) - 1 + d_G(y_1) \geq |G'|,$$

and so G' is pancyclic by Lemma 2.4. This implies pancyclicity of G, a contradiction.

Now let $h_1 = 3$. Let $G' = G - \{x_1, x_2, x_3\}$. Note that the cycle uy_1C^+vu is a hamiltonian cycle in G'. Since

$$d_{G'}(u) + d_{G'}(y_1) = d_G(u) - 1 + d_G(y_1) \geq |G'| + 1,$$

G' is pancyclic by Lemma 2.3. Hence, there are $[3, n-3]$-cycles in G. Since there is a one-chord in $C[u, v]$, G contains also $[n-1, n]$-cycles. It follows that there are no cycles of length $n-2$ in G, since we assumed that G is not pancyclic. Then obviously $vx_1 \notin E(G)$. But now, in order to avoid $\{u;, x_1, v, y_1\}$ inducing a claw with neither x_1 nor y_1 being super-heavy, $vy_1 \in E(G)$. This implies, by Claim 5.4, that $d_G(y_1) \geq (n+1)/2 - 2 + 1$ and so $d_G(u) + d_G(y_1) \geq n$. Since there is an $(n-1)$-cycle in G, G is pancyclic by Lemma 2.4, a contradiction. □

Now it follows from Claim 5.5 and Claim 5.22 that from every four consecutive vertices of H_2 at most two of them can be adjacent to u. One can easily verify that for every possible rest obtained from dividing h_2 by 4 it follows that $d_{H_2}(u) \leq \lfloor h_2/2 \rfloor + 1 \leq h_2/2 + 1$. Hence, $d_G(u) \leq h_2/2 + 3$. If $h_1 \geq 4$, then $h_2 \leq n - 6$ and we get $d_G(u) \leq n/2$, a contradiction with u being super-heavy. Hence, $h_1 \in \{2, 3\}$.

Claim 5.23. *There are no one-chords in C. Furthermore, v is adjacent to x_1.*

Proof. Note that $uy_2 \notin E(G)$, by claim 5.22. Since $y_1y_2 \in E(G)$, it follows from Claim 5.4 that $d_G(y_1) \geq (n+1)/2 - 2 + 1$, and so $d_G(u) + d_G(y_1) \geq n$. Now, if y_1 is adjacent to v, then the sum of the degrees of u and y_1 is strictly greater than n and so G is pancyclic by Lemma 1.4. Similarly, if there is a one-chord in C, then the pancyclicity of G follows from Lemma 2.4. Thus $vy_1 \notin E(G)$ and there are no one-chord in C.

To show that the second part of the claim is true, suppose the contrary, i.e., suppose that $vx_1 \notin E(G)$. Then $\{u; v, x_1, y_1\}$ induces a claw. Since neither x_1 nor y_1 is super-heavy, this contradicts G belonging to the family $\mathcal{F}(K_{1,3}, n+1)$. Hence, $vx_1 \in E(G)$. □

Claim 5.23 implies that $h_1 \neq 2$. Thus $h_1 = 3$. Since there are no one-chords in C and vx_1 is an edge in G, it follows that $\{v;\, x_1,\, x_3,\, y_{h_2}\}$ induces a claw. By Claim 5.2 none of the vertices x_1 and x_3 is super-heavy. This contradicts G being a member of the family $\mathcal{F}(K_{1,3},\, n+1)$ and completes the proof of this subcase.

Subcase 2.2: $d_{H_1}(u) \geq 2$.

Before the proof splits further into subcases, it will be shown that G does not satisfy the second of the assumptions of the theorem. Suppose the contrary, i.e., suppose that $G \in \mathcal{F}(H,\, n+1)$. From the assumptions of this subcase and from Claim 5.6 it follows that there is a triangle ux_ax_bu in G for some $x_a,\, x_b \in H_1$. If u is adjacent to some vertex $y \in H_2$ other than y_1, then, by Claim 5.4, $\{u;\, x_a,\, x_b;\, y_1,\, y\}$ induces an H. Since $G \in \mathcal{F}(H,\, n+1)$ and neither x_a nor x_b is super-heavy (by Claim 5.2), y_1 is super-heavy. But then $\{u,\, y_1\}$ is a super-heavy pair of vertices, in contradiction to Claim 5.3.

Thus assume $N_{H_2}(u) = \{y_1\}$. Since u is super-heavy and can be adjacent to at most y_1, v and all vertices of H_1, it follows that $(n+1)/2 \leq d_G(u) \leq h_1 + 2$. This, together with the choice of h_1, implies $(n-3)/2 \leq h_1 \leq (n-2)/2$. Whether n is even and equal to $2k$ or odd, and equal to $2k+1$, we get $h_1 = k-1$. In order for u to be super-heavy, its neighbourhood must be $N_G(u) = H_1 \cup \{y_1,\, v\}$, implying the existence of $[3,\, h_1+2]$-cycles in G, which can be rewritten as $[3,\, k+1]$ cycles. Note that $C' = uC^+vu$ is a cycle of length $n - h_1 = n-k+1 \leq k+2$. By appending neighbours of u along the orientation of the cycle C to C', we obtain $[k+2,\, n]$-cycles. Hence G is pancyclic, a contradiction.

This final contradiction proves that G does not belong to the family $\mathcal{F}(H,\, n+1)$. For the rest of the proof we thus assume that $G \in \mathcal{F}(D,\, n+1)$ and that $n \geq 14$.

Subcase 2.2.1: $h_1 > d_{H_1}(u) \geq 2$.

The idea of the proof of this subcase is to find a short cycle in G that can be extended by appending to it all the vertices of G, one-by-one (and thus creating all cycles of greater lengths). Firstly, the neighbourhood of u in G is examined. Then we prove that the non-neighbours of u can be used for extending the desired cycle. After the existence of short cycles in G is proved, we extend the longest of these short cycles using the observations made before and arrive at the conclusion of pancyclicity of G.

Note that the assumptions of this subcase imply $h_1 \geq 3$. Let $x_i \in N_{H_1}(u)$ be a vertex such that $x_{i+1} \in H_1$ and $ux_{i+1} \notin E(G)$.

Claim 5.24. *Suppose that u is adjacent to a super-heavy vertex $y_j \in H_2$. If $j < h_2$, then every vertex from the set $\{y_{j+1},\, \ldots,\, y_{h_2}\}$ is adjacent to y_j and not adjacent to y_1.*

Proof. Clearly, y_{j+1} is adjacent to y_j. To show that it is not adjacent to y_1, suppose the the contrary, i.e., suppose $y_1y_{j+1} \in E(G)$. Then $y_1y_{j+1}C^+uy_jC^-y_1$ is a hamiltonian cycle in G with $d_G(u) + d_G(y_j) \geq n+1$. Lemma 2.3 implies that G is pancyclic, a contradiction.

Assume $\{y_{j+1},\ ...,y_{j+m}\} \subset N_G[y_j]$ and $\{y_{j+1},\ ...,y_{j+m}\} \cap N_G(y_1) = \emptyset$ for some m satisfying $j+m < h_2$. We will show that this implies $y_j y_{j+m+1} \in E(G)$ and $y_1 y_{j+m+1} \notin E(G)$.

Suppose that y_1 is adjacent to y_{j+m+1}. Consider $G' = G - \{y_{j+1},\ ...,\ y_{j+m}\}$. Clearly, $|G'| = n - m$ and $y_1 y_{j+m+1} C^+ u y_j C^- y_1$ is a hamiltonian cycle in G'. Since none of the vertices removed from G in order to obtain G' is adjacent to y_1, it follows from Claim 5.4 that none of them is adjacent to u. Hence,

$$d_{G'}(u) + d_{G'}(y_j) = d_G(u) + d_G(y_j) - m \geq |G'| + 1,$$

and so G' is pancyclic by Lemma 2.3, implying that there are $[3,\ n-m]$-cycles in G. Note that the cycle $y_j y_{j+m} C^+ y_j$ of length $n-m+1$ can be extended to the $(n-m+2)$-cycle $y_j y_{j+m-1} y_{j+m} C^+ y_j$. Appending vertices $y_{j+m-2},\ ...,,y_{j+1}$ to this cycle, one-by-one, in the similar manner, gives $[n-m+3,\ n]$-cycles. It follows that G is pancyclic, a contradiction.

Hence, y_{j+m+1} is not adjacent to y_1 and so, by Claim 5.4, $u y_{j+m+1} \notin E(G)$. Now suppose that y_{j+m+1} is not adjacent to y_j. Then the set $\{y_1,\ u,\ y_j;\ x_i,\ x_{i+1};\ y_{j+m},\ y_{j+m+1}\}$ induces a deer in G. Since $G \in \mathcal{F}(D,\ n+1)$ and, by Claim 5.2, x_i is not super-heavy, it follows that y_1 is super-heavy. But then $\{u,\ y_1\}$ is a super-heavy pair of vertices, a contradiction with Claim 5.3. Thus $y_j y_{j+m+1} \in E(G)$. By mathematical induction the claim follows. □

Claim 5.25. *$N_{H_2}[u]$ induces a clique and at most one of the neighbours of u in H_2 is super-heavy.*

Proof. Note that it follows from Claim 5.24 and Claim 5.4 that if u is adjacent to some super-heavy vertex $y_j \in H_2 - y_{h_2}$, then $\{y_{j+1},\ ...,y_{h_2}\} \cap N_G(u) = \emptyset$. Suppose that there are two super-heavy neighbours of u in H_2, say y_j and y_m, where $j < m$. Then obviously $y_m \in \{y_{j+1},\ ...,y_{h_2}\}$, a contradiction.
Now suppose that the first part of the claim is not true. Then there are two neighbours of u, say y_a and y_b, such that $y_a y_b \notin E(G)$. But then $\{u;\ x_1,\ y_a,\ y_b\}$ induces a claw. Since x_1 is not super-heavy by Claim 5.2 and at most one vertex from the pair $\{y_a,\ y_b\}$ can be super-heavy, this contradicts G being a member of the family $\mathcal{F}(K_{1,3},\ n+1)$. □

Claim 5.26. *There are* $[3,\ 6]$*-cycles in* G.

Proof. Since $n \geq 14$ and u is super-heavy, $d_G(u) \geq 8$. Hence, u has at least four neighbours either in H_1 or else in H_2. Both $N_{H_1}[u]$ and $N_{H_2}[u]$ induce cliques, by Claim 5.6 and Claim 5.25, respectively, implying that there is an induced clique on at least five vertices in G. Thus there are $[3,\ 5]$-cycles in G.

Suppose that G is missing cycles of length six. Claims 5.6 and 5.25 imply that $d_{H_1}(u) \leq 4$ and $d_{H_2}(u) \leq 4$. Let y_j be the neighbour of u in H_2 with a highest index values. It will be first showed that $d_{H_2}(u) < 4$.

To do this, assume the contrary. Note that if there is a super-heavy vertex $y \in H_2$ that is not adjacent to u, then $|N_{H_2}(u) \cap N_{H_2}(y)| \geq 2$. Then u, y and four neighbours of u in H_2 form a cycle C_6. Thus assume that every super-heavy vertex of H_2 is a neighbour of u. It follows from Claim 5.2 and Claim 5.25 that the set of all super-heavy vertices of G is a subset of the set $\{u, v, y_j\}$. Observe that if v is not super-heavy, then it follows from the fact that $G \in \mathcal{F}(\{K_{1,3}, P_7, D\}, n+1)$ and $uy_j \in E(G)$ that G is in fact $\{K_{1,3}, P_7, D\}$-free, in contradiction with the assumptions. Thus v is super-heavy.

Now suppose that v is not adjacent to u. Since both these vertices are super-heavy, they have at least three common neighbours in G. If they share at least two neighbours in H_2, we obtain a cycle C_6, by Claim 5.25. Thus $|N_{H_2}(u) \cap N_{H_2}(v)| \leq 1$, implying that $|N_{H_1}(u) \cap N_{H_1}(v)| \geq 2$. If $d_{H_1}(u) \geq 4$, then one can create a cycle of length six using u, v and four neighbours of u from H_1, by Claim 5.6. Thus $d_{H_1}(u) \leq 3$. But then $d_G(u) \leq 7$, in contradiction with u being super-heavy and $n \geq 14$.

Hence, $uv \in E(G)$. Note that u and v have no common neighbours in H_2, since otherwise a cycle of length six can be created using u, v and four neighbours of u from H_2. Since both u and v are super-heavy, it follows that $|N_{H_1}(u) \cap N_{H_1}(v)| \geq 2$. Let x and x' be common neighbours of u and v in H_1. If y_j is adjacent to y_{h_2}, then $uy_jy_{h_2}vxx'u$ is a cycle of length six. Hence, $y_jy_{h_2} \notin E(G)$. Thus y_j is not super-heavy, either by Claim 5.24 (if $j < h_2$) or else by Claim 5.3 and the fact that v is super-heavy. Hence, the only super-heavy vertices in G are u and v. Since they are adjacent and $G \in \mathcal{F}(\{K_{1,3}, P_7, D\}, n+1)$, this implies that G is $\{K_{1,3}, P_7, D\}$-free. A contradiction.

The reasoning conducted in the previous three paragraphs proves that $d_{H_2}(u) < 4$. From the inequalities $d_G(u) \geq 8$ and $d_{H_1}(u) \leq 4$ it follows that $d_{H_1}(u) = 4$, $d_{H_2}(u) = 3$ and u is adjacent to v. Note that if $N_{H_1}(u) \cap N_{H_1}(v) \neq \emptyset$, then the existence of a cycle of length six follows from Claim 5.6 (i.e., a cycle of length six can be constructed using u, v and the four neighbours of u in H_1). Hence, u and v have no common neighbours in H_1. Let y_1, y_k and y_j be the neighbours of u in H_2 with $1 < k < j$. By Claim 5.24 neither y_1 nor y_k is super-heavy. Thus, to avoid induced claws $\{u; x_1, y_1, v\}$ and $\{u; x_1, y_k, v\}$, the vertex v is adjacent to both y_1 and y_k. Now, if there is a super-heavy vertex in H_2 that is not adjacent to u, say y, then it is adjacent to at least two neighbours of u in $H_2 \cup \{v\}$. Using u, v, y and the neighbours of u in H_2 we can then create a cycle of length six, a contradiction. It follows that the set of all super-heavy vertices of G is a subset of the set $\{u, v, y_j\}$. Note that if any of the vertices v and y_j was not super heavy, then, since both of them are adjacent to u, from the assumption of G being a member of the family $\mathcal{F}(\{K_{1,3}, P_7, D\}, n+1)$ it follows that G is in fact $\{K_{1,3}, P_7, D\}$-free. A contradiction. Hence, both y_j and v are super heavy. This implies that $j < h_2$, by Lemma 2.3, and so $y_jy_{h_2} \in E(G)$, by Claim 5.24. But now $uy_1y_ky_jy_{h_2}vu$ is a cycle of length six in G. This final contradiction completes the proof of Claim 5.26. □

Claim 5.27. *Let $A = \{x_{a+1}, ..., x_{a+p}\} \subset H_1$ be a maximal set of consecutive non-neighbours of u in H_1 (i.e., $x_a \in N_{H_1}(u)$ and either $x_{a+p+1} \in N_{H_1}(u)$ or else $x_{a+p+1} = v$). Then $A \subset N_G(x_a)$.*

Proof. Since the statement is obvious for $p = 1$, assume $p \geq 2$. Suppose that the claim is not true. Then there is a vertex $x_{a+j} \in A$ adjacent to x_a such that $1 < j < p - 1$ and $x_ax_{a+j+1} \notin E(G)$. We divide the proof of this claim into three subclaims.

Claim 6.27.1. *Let $B = \{y_{b+1}, ..., y_{b+q}\} \subset H_2$ be a maximal set of consecutive non-neighbours of u in H_2 (i.e., $y_b \in N_{H_2}(u)$ and either $y_{b+q+1} \in N_{H_2}(u)$ or else $y_{b+q+1} = v$). Then $B \subset N_G(y_b)$.*

Proof. Again, assume $q \geq 2$, since the statement is obviously true for $q = 1$, and suppose it is not true. Then there are vertices $y_{b+l}, y_{b+l+1} \in B$ such that $y_b y_{b+l} \in E(G)$ and $y_b y_{b+l+1} \notin E(G)$. But now $\{x_{a+j+1}, x_{a+j}, x_a, u, y_b, y_{b+l}, y_{b+l+1}\}$ induces P_7. Since neither x_{a+j+1} nor x_a is super-heavy, this contradicts G belonging to the family $\mathcal{F}(P_7, n+1)$. □

Claim 6.27.2. $d_{H_1}(u, x_{h_1}) = 3$.

Proof. Suppose the statement is not true. First assume $d_{H_1}(u, x_{h_1}) \geq 4$. Then there is an induced path P_5 in H_1 connecting u with x_{h_1}, say $uxx'x''x_{h_1}$. Recall that $y_k \in N_{H_2}(u)$ is a vertex such that $y_{k+1} \in H_2$ and $uy_{k+1} \notin E(G)$. It follows that $\{x_{h_1}, x'', x', x, u, y_k, y_{k+1}\}$ induces a P_7, a contradiction with G being a member of the family $\mathcal{F}(P_7, n+1)$ (by Claim 5.2).

Now assume $d_{H_1}(u, x_{h_1}) \leq 2$. First we note that whether or not u is adjacent to x_{h_1}, there is a vertex $x \in H_1$ such that uxx_{h_1} is a path P_3 (not necessarily an induced one). It is obviously true when $ux_{h_1} \notin E(G)$; if the opposite is true, it follows from Claim 5.6 and the fact that $d_{H_1}(u) \geq 2$.

Furthermore, the same is true for y_{h_2}: whether or not this vertex is adjacent to u, there is $y \in H_2$ such that uyy_{h_2} is a path P_3. If $uy_{h_2} \in E(G)$, it follows from Claim 5.25 for $y = y_1$. Otherwise it is a corollary from the Claim 6.27.1.

Hence, $uyy_{h_2}vx_{h_1}xu$ is a cycle of length six. Since neighbours of u in H_2 induce a clique, by Claim 5.25, they can be appended to this cycle one-by-one between u and y, creating at least $[6, d_{H_2}(u)+4]$-cycles. Consider the longest cycle of those just obtained. By Claim 5.6, the neighbours of u from H_1 can be added to this cycle in a similar manner. Finally the vertices from the gaps between the neighbours of u in $C[y_1, y_{h_2}]$ can be appended to this cycle (again, one-by-one), due to the Claim 6.27.1. In this way we obtain $[6, h_2+d_{H_1}(u)+2]$-cycles.

Note that $uyy_{h_2}C^+u$ is a cycle of length $n - h_2 + 2$. To this cycle we also can append all vertices from H_2, in the way described above, thus obtaining $[n - h_2 + 2, n]$-cycles. Since G is not pancyclic and it contains $[3, 6]$- (by Claim 5.25) and $[6, h_2 + d_{H_1}(u) + 2]$-cycles, it must be

$$h_2 + d_{H_1}(u) + 2 < n - h_2 + 2 = h_1 + 4 \leq h_2 + 4,$$

implying $d_{H_1}(u) < 2$. This contradicts the assumptions of this subcase. The claim follows. □

Claim 6.27.3. *There are vertices $y \in H_2$ and $x, x' \in H_1$ such that $uyy_{h_2}vx_{h_1}x'xu$ is a cycle in G.*

Proof. Clearly, since $d_{H_1}(u, x_{h_1}) = 3$, there are vertices $x, x' \in H_1$ such that $uxx'x_{h_1}$ is a path P_4. Now, if $uy_{h_2} \in E(G)$, then, by Claim 5.25, there is a path $uy_1y_{h_2}$ and we can set

$y = y_1$. Otherwise let y be the last (i.e. with the highest index) neighbour of u in H_2. It is adjacent to y_{h_2} by Claim 6.27.1, and so $uyy_{h_2}vx_{h_1}x'xu$ is a demanded cycle. □

By Claims 5.26 and 6.27.3 there are [3, 7]-cycles in G. Consider now the cycle $C' = uyy_{h_2}vx_{h_1}x'xu$. We can extend C' by appending to it, one-by-one, vertices from $N_{H_2}(u)$ (by Claim 5.25), then the remaining vertices from H_2 (by Claim 6.27.1) and finally all neighbours of u from H_1 (by Claim 5.6). In this way we obtain $[7,\ h_2 + d_{H_1}(u) + 4]$-cycles.
Note that $uyy_{h_2}C^+u$ is a cycle of length $h_1 + 4$. This cycle also can be extended with vertices from $N_{H_2}(u)$ and then the remaining vertices from H_2. This procedure gives $[h_1 + 4,\ n]$-cycles.
Since G is not pancyclic, it must be $h_2 + d_{H_1}(u) + 4 < h_1 + 4$. But by the choice of h_1 we have also $h_1 \leq h_2$. These inequalities imply that $d_{H_1}(u) < 0$, an obvious contradiction. □

Claim 5.28. *Let $A = \{y_{a+1},\ ...,\ y_{a+p}\}$ be a set of consecutive non-neighbours of u in H_2 such that $uy_a \in E(G)$ and $y_a y_{a+p+1} \in E(G)$ (where we assume $y_{h_2+1} = v$). Let $P = v_1v_2...v_m$ be a path with $m \geq 3$, $v_1 = y_a$, $v_m = y_{a+p+1}$ and $v_i \in A$ for $i = 2,\ ...,\ m-1$. Finally, let C' be a cycle of length q in G such that $u,\ v \in V(C')$, $C'[v,\ u] = \{v,\ x_{h_1},\ x_{h_1-1},\ ...,\ x_1,\ u\}$, $A \cap V(C') = \emptyset$ and $y_a y_{a+p+1}$ is an edge of C'.*
Then one can obtain $[q + 1,\ q + m - 2]$-cycles by appending some of the vertices from the path P to the cycle C' and omitting at most one vertex from $V(C')$.

Proof. If y_a is super-heavy, it is adjacent to every vertex from A, by Claim 5.24, and so the statement follows. Now assume that y_a is not super-heavy.

First we show that there is a vertex in $V(C')$ the omitting of which along C' results in a cycle of length $q - 1$. Clearly, if $ux_2 \in E(G)$, then x_1 is such a vertex (namely, the cycle of length $q - 1$ is $x_2uC'^+x_2$). If $ux_2 \notin E(G)$, then $x_1x_3 \in E(G)$ (it follows from Claim 5.6 if $ux_3 \in E(G)$, or from Claim 5.27 if $ux_3 \notin E(G)$) and the vertex that can be omitted is x_2.

The proof is by induction with respect to m. For $m = 3$ we need to point out only a cycle of length $q+1$. Obviously, $uC'^+y_av_2y_{a+p+1}C'^+u$ is such a cycle. For the case when $m = 4$ we want to find cycles of lengths $q + 1$ and $q + 2$. The previous is $uC'^+y_av_2v_3y_{a+p+1}C'^+\hat{x}C'^+u$ (where $\hat{x}$ stands for omitting either x_1 or else x_2) and the latter is $uC'^+y_av_2v_3y_{a+p+1}C'^+u$.

Now assume the statement is true for some fixed $m \geq 4$, as well as for $m - 1$. Consider a path $P = v_1...v_{m+1}$ satisfying the assumptions. In order to avoid $\{x_{i+1},\ x_i,\ u,\ y_a,\ v_2,\ v_3,\ v_4\}$ inducing a P_7 with neither x_i nor y_a being super-heavy, there must be one of the edges y_av_3, y_av_4 or v_2v_4.

If $y_av_3 \in E(G)$ (or $v_2v_4 \in E(G)$), $P' = y_av_3P^+y_{a+p+1}$ (or $P' = y_av_2v_4P^+y_{a+p+1}$) is a path on m vertices that allows us to obtain $[q + 1,\ q + m - 2]$-cycles. In order to obtain a cycle of length $q + m - 1$, we simply append all vertices from P to C' (i.e., this cycle is $uC'^+y_av_2...v_my_{a+p+1}C'^+u$).

If there is an edge y_av_4, it creates a path $P' = y_av_4P^+y_{a+p+1}$ on $m-1$ vertices, and so there are $[q+1,\ q+m-3]$-cycles. To obtain a cycle of length $q+m-1$, simply append all vertices from P to C'. Finally, omitting x_1 or x_2 in this last cycle creates a $(q + m - 2)$-cycle. □

So far we know the structure of u neighbourhoods in H_1 and H_2 and the parts of the cycle C that lie between u's neighbours. To describe the remaining part of C, let y_j denote the last (i.e. the one with the highest index) neighbour of u in H_2.

Claim 5.29. *$y_j \neq y_{h_2}$ and $y_j y_{h_2} \notin E(G)$.*

Proof. Suppose that the statement is not true. Then, by Claim 5.27 and the fact that $d_{H_1}(u) \geq 2$, there is a cycle $uy_{h_2}vx_{h_1}xu$ (if $y_j = y_{h_2}$) of length five or a cycle $uy_jy_{h_2}vx_{h_1}xu$ (if $y_jy_{h_2} \in E(G)$) of length six. Since neighbours of u in H_1 induce a clique, by Claim 5.6, they can be appended to this cycle, one-by-one. Then the same can be done with the remaining vertices from H_1, by Claim 5.27, and subsequently with neighbours of u from H_2, as they also induce a clique, by Claim 5.25.

In this manner we obtain at least $[6,\, h_1 + d_{H_2}(u) + 2]$-cycles, the longest of which contains all vertices from G but the non-neighbours of u in H_2. Denote this longest cycle C'. These remaining vertices can be divided into disjoint maximal sets of consecutive non-neighbours of u along C. Applying Claim 5.28 to C' with the first of these sets as A (where the path P from Claim 5.28 consists of all vertices from A), gives a cycle C'' with $V(C'') = V(C') \cup A$, and every cycle shorter than C''. Applying Claim 5.28 to C'' and the remaining sets of non-neighbours of u, one-by-one, we finally arrive at the cycle C. Since this procedure guarantees creating cycles of all lengths from $h_1 + d_{H_2}(u) + 2$ up to n, there are $[6,\, n]$-cycles in G. Since there are also $[3,\, 5]$-cycles, by Claim 5.26, G is pancyclic, a contradiction. □

Note that if y_j was super-heavy, it would be adjacent to y_{h_2} by Claim 5.24. Hence it follows from Claim 5.29 that y_j is not super-heavy.

Claim 5.30. *Let y_m be the last neighbour (i.e., with the highest index) of y_j in $C[y_j,\, y_{h_2}]$. Then $y_my \in E(G)$ for $y \in \{y_{m+1},\, \ldots,\, y_{h_2}\}$.*

Proof. Note that $m \leq h_2 - 1$ by Claim 5.29. Since the statement is obvious for $m = h_2 - 1$, assume $m \leq h_2 - 2$. Suppose that the claim is not true. Then there is some vertex $y_b \in \{y_{m+1},\, \ldots,\, y_{h_2-1}\}$ such that $y_by_m \in E(G)$ and $y_my_{b+1} \notin E(G)$. But then $\{x_{i+1},\, x_i,\, u,\, y_j,\, y_m,\, y_b,\, y_{b+1}\}$ induces a P_7 with neither x_i nor y_j being super-heavy. A contradiction. □

Now it follows from Claims 5.27, 5.29 and 5.30 and the fact that $d_{H_1}(u) \geq 2$ that there is a cycle $C' = uy_jy_my_{h_2}vx_{h_1}xu$, where x is the neighbour of u in H_1 with the highest index if $ux_{h_1} \notin E(G)$ and $x = x_1$ otherwise. To this cycle C_7 we can append neighbours of u, one-by-one, by Claim 5.6 and Claim 5.25 and then the non-neighbours of u from H_1, by Claim 5.27. Vertices from the set $\{y_{m+1},\, \ldots,\, y_{h_2-1}\}$ can then be added to the cycle due to Claim 5.30. Finally, Claim 5.28 allows us to extend the longest of just created cycles using the non-neighbours of u in H_2 (just like in the proof of Claim 5.29) up to the hamiltonian cycle C. Hence, there are $[7,\, n]$-cycles in G. Together with Claim 5.26 this implies that G is pancyclic. This contradiction completes the proof of this subcase.

Subcase 2.2.2: $h_1 \geq 2$, $d_{H_1}(u) = h_1$.

The idea of the proof in this subcase is the same as in the previous one. The neighbourhood of u in G is firstly examined. Then we focus on the parts of the cycle C that lie between the neighbours of u (i.e., the parts in which u has no neighbours) and show that the vertices lying on these parts can be used to extend some cycles. The next step is to prove the existence of short cycles in G. Finally, we choose a specific short cycle of G, and using the previous observation extend it by appending all vertices of G to it, one-by-one, up to the cycle C.

Claim 5.31. *None of the neighbours of u in H_2 is super-heavy.*

Proof. Assume the contrary. Then u is adjacent to some super-heavy vertex $y_j \in H_2$. Note that $j \geq 3$, by Claim 5.3, $y_{j-1}y_{j+1} \notin E(G)$, by Lemma 2.1, and $y_1y_j \in E(G)$, by Claim 5.4. Furthermore, it must be $y_1y_{j+1} \notin E(G)$ (where we assume $y_{h_2+1} = v$), since otherwise $C' = y_1y_{j+1}C^+uy_jC^-y_1$ would be a hamiltonian cycle in G with $d_{C'}(u, y_j) = 1$ and $d_G(u) + d_G(y_j) \geq n + 1$, and thus G would be pancyclic by Lemma 2.3.

Claim 5.3 implies that neither y_{j-1} nor y_{j+1} is super-heavy. Since $G \in \mathcal{F}(K_{1,3}, n+1)$, it follows that $\{y_j;\, u, y_{j-1}, y_{j+1}\}$ cannot induce a claw. Hence, u is adjacent to y_{j-1} or y_{j+1}.

Suppose $uy_{j+1} \in E(G)$. Since $y_1y_{j+1} \notin E(G)$, Claim 5.4 implies that $y_{j+1} \notin H_2$ and so $j = h_2$ and $y_{j+1} = v$. Consider $G' = G - H_1$. G' is obviously hamiltonian with the cycle $C' = y_jvuC^+y_j$ being its hamiltonian cycle. Since

$$d_{G'}(u) + d_{G'}(y_j) \geq (n + 1)/2 - h_1 + (n + 1)/2 \geq |G'| + 1,$$

G' is pancyclic by Lemma 2.5. Appending vertices from H_1 to C', one-by-one, creates cycles of all lengths greater than $|G'|$ and so G is also pancyclic, a contradiction. Hence, $uy_{j+1} \notin E(G)$ and $uy_{j-1} \in E(G)$.

Suppose that $uv \notin E(G)$. Consider $G' = G - \{x_1, ..., x_{h_1-1}\}$, a hamiltonian graph with a hamiltonian cycle $C' = y_1y_jC^+x_{h_1}uy_{j-1}C^-y_1$. First it will be shown that G' is pancyclic. Indeed, if $uy_2 \notin E(G)$, then $y_2 \in N_G(y_1) \setminus N_G(u)$ and Claim 5.4 together with the fact that $uv \notin E(G)$ imply $d_G(y_1) \geq (n + 1)/2 - h_1 + 1$. Hence, $d_{G'}(y_1) + d_{G'}(y_j) \geq |G'| + 1$, and pancyclicity of G' follows by Lemma 2.3. If $uy_2 \in E(G)$, then a similar argument leads to the inequality $d_{G'}(y_1) + d_{G'}(y_j) \geq |G'|$. This inequality together with the cycle $uy_2C'^+u$ of length $|G'| - 1$ implies that G' is pancyclic by Lemma 2.4. It follows that there are $[3, |G'|]$-cycles in G. Since the vertices from H_1 can be appended to the cycle C' one-by-one, thus creating $[|G'|, n]$-cycles, G is pancyclic, a contradiction.

Hence, $uv \in E(G)$. Consider $G' = G - H_1$ with a Hamilton cycle $C' = y_1y_jC^+vuy_{j-1}C^-y_1$. Again, depending on wether or not u is adjacent to y_2, we have $d_G(y_1) \geq (n+1)/2 - h_1 - 1$ (if it is) or $d_G(y_1) \geq (n+1)/2 - h_1$. In the previous case $uy_2C'^+u$ is a $(|G'| - 1)$-cycle in G' and the inequality $d_{G'}(y_1) + d_{G'}(y_j) \geq |G'|$ holds, implying that G' is pancyclic by Lemma 2.4. In the latter case we have $d_{G'}(y_1) + d_{G'}(y_j) \geq |G'| + 1$ and so G' is pancyclic by Lemma 2.3. Again, pancyclicity of G' implies pancyclicity of G, since the vertices from H_1 can be appendend to C' one-by-one. Thus G is pancyclic, a contradiction. □

Claim 5.32. $N_{H_2}[u]$ *induces a clique in* G.

Proof. Suppose the claim is not true, i.e. that there are vertices y_a, $y_b \in N_{H_2}(u)$ such that $y_a y_b \notin E(G)$. Then $\{u;\, y_a,\, y_b,\, x_1\}$ induces a claw. Since neither y_a nor y_b is super-heavy, by Claim 5.31, this contradicts G being a member of the family $\mathcal{F}(K_{1,3},\, n+1)$. □

Claim 5.33. *There are* $[3,\, 5]$*-cycles in* G.

Proof. Since u is super-heavy and $n \geq 14$, we have $d_G(u) \geq 8$. Hence, u has at least four neighbours in H_1 or H_2. Both $N_{H_1}[u]$ and $N_{H_2}[u]$ are complete subgraphs of G, by Claim 5.6 and 5.32, respectively, and so the claim follows. □

Claim 5.34. *Let* $A = \{y_{a+1},\, ...,\, y_{a+p}\}$ *be a set of consecutive non-neighbours of* u *in* H_2 *such that* $uy_a \in E(G)$ *and* $y_a y_{a+p+1} \in E(G)$ *(where we assume* $y_{h_2+1} = v$*). Let* $C' = uC^+ y_a y_{a+p+1} C^+ u$ *be a cycle of length* $q = n - p$*. Finally, let* $P = v_1 v_2 ... v_m$ *be a path with* $m \geq 3$*,* $v_1 = y_a$*,* $v_m = y_{a+p+1}$ *and* $v_i \in A$ *for* $i = 2,\, ...,\, m-1$.

Then one can obtain $[q+1,\, q+m-2]$*-cycles by appending some of the vertices from the path* P *to the cycle* C' *and omitting at most two neighbours of* u *belonging to* $V(C')$.

Proof. The proof is by induction on m. For the case when $m = 3$ we only need to point out a cycle of length $q+1$. It is easy to see that $y_a v_2 y_{a+p+1} C'^+ y_a$ is such a cycle.

Assume $m = 4$. By the assumptions of this subcase u is adjacent to x_2 and so $y_a v_2 v_3 y_{a+p+1} C'^+ x_2 u C'^+ y_a$ is a cycle of length $q+1$. Append x_2 to this cycle in order to obtain a cycle with $q+2$ vertices.

Now let $m = 5$. Clearly, the cycle $C'' = y_a v_2 v_3 v_4 y_{a+p+1} C'^+ y_a$ has length $q+3$. Using the edge ux_2 to omit vertex x_1 we obtain a cycle of length $q+2$. If $h_1 \geq 3$, then the chord ux_3 in the cycle C'' creates a cycle of length $q+1$. Otherwise $h_1 = 2$. Now, if u is adjacent to v, then the edge uv is a two-chord in C'', and so there is a $(q+1)$-cycle in G. If $uv \notin E(G)$ and $uy_2 \notin E(G)$, it follows from Claim 5.4 that $d_G(y_1) \geq (n+1)/2 - 1$ and so $d_G(u) + d_G(y_1) \geq n$. Since the existence of a cycle of length $n-1$ in G follows from the assumptions of this subcase, this contradicts Lemma 2.4. Finally, if $uv \notin E(G)$ and $uy_2 \in E(G)$, then $uy_2 C'^+ y_a v_2 v_3 v_4 y_{a+p+1} C'^+ x_2 u$ is a cycle of length $q+1$.

Assume that the claim is true for some $m \geq 5$ and consider a path P of order $m+1$ that satisfies the assumptions. If $\{x_1,\, u,\, y_a,\, v_2,\, v_3,\, v_4,\, v_5\}$ induces a P_7, this contradicts G belonging to the family $\mathcal{F}(P_7,\, n+1)$, since neither x_1 nor y_a is super-heavy (by Claims 5.2 and 5.31). Hence, there is an edge in $G[\{y_a,\, v_2,\, v_3,\, v_4,\, v_5\}]$ that does not belong to the path P. This edge creates a shorter path, of length at least $m-2$, that satisfies the assumptions of the Claim. It follows that we can obtain $[q+1,\, q+m-4]$-cycles in the desired manner. Obviously, $C'' = y_a P^+ y_{a+p+1} C'^+ y_a$ is a cycle of length $q+m-1$. To obtain cycles of lengths $q+m-3$ and $q+m-2$ use chords of C'' as described in the case of $m = 5$. □

From now on let y_j denote the neighbour of u in H_2 with the highest index.

Claim 5.35. $j \leq h_2 - 3$ *and* y_j *is adjacent neither to* y_{h_2} *nor* y_{h_2-1}.

Proof. Suppose the first part of the Claim is not true. Then $j \in \{h_2-2,\, h_2-1,\, h_2\}$ and there is one of the cycles $uy_{h_2-2}y_{h_2-1}y_{h_2}vx_{h_1}u$, $uy_{h_2-1}y_{h_2}vx_{h_1}u$ or $uy_{h_2}vx_{h_1}u$ in G. Let C' denote that cycle. Neighbours of u both in H_1 and in H_2 induce cliques (by Claims 5.6 and 5.32, respectively), and so they can be appended to C', one-by-one. Let C'' be the cycle C' with all neighbours of u appended to it. The remaining vertices are the non-neighbours of u in H_2. Let $\{y^1,\, ...,\, y^{d_{H_2}(u)}\} \subset H_2$ be the neighbours of u in H_2 sorted by their indices in ascending order. Applying Claim 5.34 to the cycle C'' and the set $A = C[y^1,\, y^2]$ we obtain cycles longer than C'' up to the cycle $C''' = y^1C^+y^2C''^+y^1$. Now we can apply Claim 5.34 to the cycle C''' and the set $C[y^2,\, y^3]$. Repeating this procedure up to the set $C[y^{d_{H_2}(u)-1},\, y^{d_{H_2}(u)}]$, we finally arrive at the cycle C. It follows that there are $[|C'|,\, n]$-cycles in G. Since $|C'| \leq 6$, together with Claim 5.33 this implies that G is pancyclic, a contradiction.

If y_j is adjacent to either y_{h_2-1} or y_{h_2}, the similar argument as presented above applied to the cycle $uy_jy_{h_2-1}y_{h_2}vx_{h_1}u$ or $uy_jy_{h_2}vx_{h_1}u$ leads to the pancyclicity of G, contradicting our assumptions. Note that Claim 5.34 can be also applied to the sets $A = \{y_{j+1},\, ...,\, y_{h_2-2}\}$ and $A = \{y_{j+1},\, ...,\, y_{h_2-1}\}$. □

Consider now the neighbour of y_j in H_2 with the highest index. Let y_m denote this vertex. It follows from Claim 5.35 that $m \leq h_2 - 2$ and so it makes sense to consider also the neighbour of y_m in H_2 with the highest index, say $y_{m'} \in H_2$. Note that the choice of j, m and m' implies that the path $x_{h_1}uy_jy_my_{m'}$ is an induced one.

Claim 5.36. *$y_{m'}y \in E(G)$ for every $y \in C[y_{m'+1},\, y_{h_2}]$.*

Proof. Assume the contrary and let $G' = G[C[y_{m'},\, y_{h_2}]]$. It follows that there are vertices y', $y'' \in C[y_{m'},\, y_{h_2}]$ such that the set $\{y_{m'},\, y',\, y''\}$ induces P_3. By the choice of y', y'', j, m and m' it follows that $x_{h_1}uy_jy_my_{m'}y'y''$ is an induced path P_7. Since neither x_{h_1} nor y_j is super-heavy, by Claims 5.2 and 5.31, this contradicts G belonging to the family $\mathcal{F}(P_7,\, n+1)$. □

Claim 5.37. *Assume that the cycle $C' = y_my_{m'}y_{h_2}C^+y_m$ has length q. Let $P = v_1...v_l$ be a path with $l \geq 3$, $v_1 = y_m$, $v_l = y_{m'}$ and $v_i \in C[y_m,\, y_{m'}]$ for $i = 2,\, ...,\, l-1$.*

Then one can obtain $[q+1,\, q+l-2]$-cycles by appending some of the vertices from P to C' and omitting at most x_1.

Proof. Since the Claim is obviously true for $l = 3$, consider $l = 4$. Then $y_mv_2v_3y_{m'}C^+y_m$ is a cycle of length $q+2$ and $y_mv_2v_3y_{m'}C^+x_2uC^+y_m$ is a cycle of length $q+1$.

For the proof by induction assume that the statement is true for some fixed $l \geq 4$ and for $l-1$. Consider now a path $P = v_1...v_{l+1}$ satisfying the assumptions of the Claim. Since $G \in \mathcal{F}(P_7,\, n+1)$ and neither x_1 nor y_j is super-heavy (by Claims 5.2 and 5.31), the set $\{x_1,\, u,\, y_j,\, y_m,\, v_2,\, v_3,\, v_4\}$ cannot induce a P_7. Note that by the choice of j and m both u and y_j have no neighbours in the set $C[y_{m+1},\, y_{m'}]$. It follows that there exists an edge in $G[\{y_m,\, v_2,\, v_3,\, v_4\}]$ that does not belong to the path P. This edge, say $v'v''$, creates a path $P' = y_mP^+v'v''P^+y_{m'}$ of length at most l or $l-1$. By the induction hypothesis there are $[q+1,\, q+l-3]$-cycles in G, created in the manner desired. Obviously, the cycle $y_mP^+y_{m'}C^+y_m$ has length $q+l-1$ and the cycle $y_mP^+y_{m'}C^+x_2uC^+y_m$ has length $q+l-2$. By mathematical induction the claim is true. □

Claim 5.38. *There are* $[7, n]$*-cycles in* G.

Proof. Claim 5.36 implies that $y_{m'}y_{h_2} \in E(G)$. Hence, $C' = uy_jy_my_{m'}y_{h_2}vx_{h_1}u$ is a cycle C_7. Let $\{y^1, ..., y^{d_{H_2}(u)}\}$ denote the neighbours of u in H_2 sorted by their indices in ascending order.

Just as in the proof of Claim 5.35 we can extend the cycle C' by appending to it all neighbours of u (since $N_{H_1}[u]$ and $N_{H_2}[u]$ induce cliques in G) and then all non-neighbours of u that belong to one of the sets $C[y^l, y^{l+1}]$ for $l \in \{1, ..., d_{H_2}(u)-1\}$ or to the set $C[y_j, y_m]$ (by Claim 5.34), as well as those belonging to the set $C[y_{m+1}, y_{m'-1}]$ (by Claim 5.36). To the longest of just created cycles, that is the cycle $y_{h_2}C^+y_{m'}y_{h_2}$, we can then add all vertices from the set $C[y_{m'+1}, y_{h_2}]$, also one-by-one, by Claim 5.37, thus arriving finally at the cycle C. □

It follows from Claims 5.33 and 5.38 that G is missing only cycles of length six. Since the cycle $C' = uy_jy_my_{m'}y_{h_2}vx_{h_1}u$ is of length seven, it follows that $uv, y_{m'}v \notin E(G)$.

Claim 5.39. C' *is an induced cycle.*

Proof. To prove this fact we need to show that $vy_m, vy_j \notin E(G)$ (by the choice of j, m, m' and the fact that v is adjacent neither to u nor to $y_{m'}$). If $vy_m \in E(G)$, then $vy_my_jux_1x_{h_1}v$ is a cycle C_6 (since $d_{H_1}(u) \geq 2$ and $N_{H_1}[u]$ induces a clique). Since $n \geq 14$, $uv \notin E(G)$ and u is super-heavy, it follows that u has at least four neighbours in H_1 or H_2. If $vy_j \in E(G)$, these neighbours can be used to obtain a cycle C_6 from the cycle $uy_jvx_{h_1}x_1u$. Hence, the claim holds. □

Claim 5.40. $h_1 \leq 3$.

Proof. First observe that if some vertex $x \in H_1$ is not adjacent to v, then it follows from the assumptions of this subcase and Claim 5.39 that the path $xuy_jy_my_{m'}y_{h_2}v$ is an induced one. Since neither x nor y_j is super-heavy, by Claim 5.31 and Claim 5.6, respectively, this contradicts G being a member of the family $\mathcal{F}(P_7, n+1)$. Hence, $N_{H_1}(v) = N_{H_1}(u) = H_1$. Now suppose that the claim is not true, i.e., suppose $h_1 \geq 4$. Since the neighbours of u in H_1 induce a clique, by Claim 5.6, and they are adjacent to v by the previous observation, it follows that four of them together with u and v form a cycle C_6. A contradiction. □

Since $n \geq 14$, u is super-heavy and $uv \notin E(G)$, Claim 5.40 implies that $d_{H_2}(u) \geq 5$. But then the subgraph of G induced by $N_{H_2}[u]$ is a clique of order at least six. Hence, there is a cycle of length six in G. This final contradiction completes the proof.

□

6 Proof of Theorem 1.32

For the convenience of the reader, we restate Theorem 1.32 below.

Theorem 1.32 (WW [48]) *Let G be a 2-connected graph which is not a cycle and let S be a connected graph with $S \neq P_3$. Then G being claw-o_1-heavy and S-c_1-heavy implies G is pancyclic if and only if $S = P_4$, P_5, Z_1 or Z_2.*

Note that every P_4-c_1-heavy graph is P_5-c_1-heavy and that every P_5-c_1-heavy graph is P_5-o_1-heavy. Furthermore, we notice that every Z_2-c_1-heavy graph is Z_2-o_1-heavy. Thus Theorem 1.29 implies the following.

Corollary 6.1. *Let G be a 2-connected, claw-o_1-heavy graph that is not a cycle. If G is S-c_1-heavy, where S is one of P_4, P_5 or Z_2, then G is pancyclic.*

Observe that every claw-free and S-free graph is claw-o_1-heavy and S-c_1-heavy. By Theorem 1.11 the only graphs S such that every $\{K_{1,3}, S\}$-free graph is pancyclic are P_4, P_5, Z_1 or Z_2. This proves the 'only if' part of Theorem 1.32. In order to complete the proof of Theorem 1.32 we only need to show that every claw-o_1-heavy and Z_1-c_1-heavy graph other than a cycle is pancyclic.

Let $A = \{v_1, v_2, v_3, v_4\}$ be a subset of vertices of G. If $G[A]$ is isomorphic to Z_1, with the set of its edges being $\{v_1v_2, v_2v_3, v_3v_1, v_3v_4\}$, we say that $\{v_1, v_2; v_3, v_4\}$ induces a Z_1. Note that if $\{v_1, v_2; v_3, v_4\}$ induces a Z_1 in a Z_1-c_1-heavy graph G, then at least one of the vertices v_1 and v_2 is super-heavy.

The following Lemma gives some information about the structure of claw-o_1-heavy graphs. Its proof was presented in [35]. We include it for completeness.

Lemma 6.1. *Let G be a 2-connected, claw-o_1-heavy graph of order n and let $r, s \in V(G)$ be vertices such that $G - \{r, s\}$ is not connected. Then*

1. *$G - \{r, s\}$ has exactly two components,*
2. *for any distinct neighbours x and x' of r (s) belonging to the same component of $G - \{r, s\}$ either $xx' \in E(G)$ or else $xx' \notin E(G)$ and $d_G(x) + d_G(x') \geq n + 1$.*

Proof. We begin with a simple observation: if two non-adjacent vertices x and y of G have no more than two common neighbours, then $d_G(x) + d_G(y) \leq (n - 2) + 2 = n$. Now for the proof of *1.* assume that G_1, G_2 and G_3 are three of the components of $G - \{r, s\}$. Let x_1, x_2 and x_3 be neighbours of r in G_1, G_2 and G_3, respectively. Since x_1 and x_2 are not adjacent and they have at most two common neighbours, namely r and s, it follows from the previous observation that $d_G(x_1) + d_G(x_2) \leq n$. Similarly, $d_G(x_1) + d_G(x_3) \leq n$ and $d_G(x_2) + d_G(x_3) \leq n$. Since $\{r; x_1, x_2, x_3\}$ induces a claw in G, this contradicts G being claw-o_1-heavy. Thus $G - \{r, s\}$ has exactly two components.

Let x and x' be neighbours of r belonging to the same component of $G - \{r, s\}$. Recall that for any neighbour x'' of r from the other component we have $d_G(x) + d_G(x'') \leq n$ and

$d_G(x') + d_G(x'') \leq n$. Assume that x and x' are not adjacent. Since the set $\{r;\, x,\, x',\, x''\}$ induces a claw in G and G is claw-o_1-heavy, it follows from the previous observation that $d_G(x) + d_G(x') \geq n+1$. Thus *2.* holds. ☐

Now we are ready to present the proof of Theorem 1.32.

Proof of Theorem 1.32: Theorem 1.32 will be proved by contradiction. Suppose that a graph G of order n satisfies the assumptions of the theorem but is not pancyclic. By Corollary 6.1 we assume that G is claw-o_1-heavy and Z_1-c_1-heavy. It follows from Theorem 1.11 that there is either an induced claw or an induced Z_1 in G, implying that there is a super heavy vertex $u \in V(G)$. Consider $G' = G - u$. Note that G' is claw-o-heavy and Z_1-c-heavy. If G' is two-connected, then it is hamiltonian by Theorem 1.31 and so G is pancyclic by Lemma 2.1, a contradiction. Hence, there is a vertex $v \in V(G)$ such that $G - \{u, v\}$ is not connected. Lemma 6.1 implies that $G - \{u, v\}$ consists of exactly two components. Note that G is hamiltonian by Theorem 1.31. Let $C = uy_1...y_{h_2}vx_{h_1}...x_1u$ be a hamiltonian cycle in G, where $H_1 = \{x_1, ..., x_{h_1}\}$ and $H_2 = \{y_1, ..., y_{h_2}\}$ are the components of $G - \{u, v\}$. Without loss of generality assume $h_1 \leq h_2$.

First we provide some information about H_1.

Claim 6.1. *There are no super-heavy vertices in H_1.*

Proof. Consider a vertex $x \in H_1$. Clearly, its neighbourhood is a subset of the set $(H_1 - x) \cup \{u, v\}$. Since $h_1 \leq h_2$, we have $h_1 \leq (n-2)/2$ and so $d_G(x) \leq n/2$. ☐

With the next two claims we establish all information about the neighbourhood of u in G that is needed to complete the proof.

Claim 6.2. *$N_{H_1}[u]$ induces a clique in G.*

Proof. Since the statement is obvious for $h_1 = 1$ and $h_1 = 2$, assume $h_1 \geq 3$. By Claim 6.1 there are no two vertices in H_1 with sum of degrees greater than n. The Claim follows from Lemma 6.1. ☐

Claim 6.3. *Every neighbour of u in H_2 other than y_1 is super-heavy.*

Proof. Let y be a neighbour of u in H_2 other than y_1. Note that y_1 is not super-heavy, since otherwise $d_G(u) + d_G(y_1) \geq n+1$ and G would be pancyclic by Lemma 1.4. First assume that $y_1y \notin E(G)$. It follows from Lemma 6.1 that $d_G(y_1) + d_G(y) \geq n+1$. Since y_1 is not super-heavy, in order for this inequality to be satisfied y must be super-heavy.

Now assume that y is adjacent to y_1. Since $\{y, y_1;\, u, x_1\}$ induces a Z_1 and G is Z_1-c_1-heavy, it follows that y is a super-heavy vertex. ☐

Note that if u has at least two neighbours in H_1, then $\{x, x'; u, y_1\}$ induces Z_1 for any two $x, x' \in N_{H_1}(u)$, by Claim 6.2. Since G is Z_1-c$_1$-heavy, either x or x' is super-heavy. This contradicts Claim 6.1. Thus $d_{H_1}(u) = 1$. This implies that $d_{H_2}(u) \geq (n-3)/2$. Since there are at most $(n-3)$ vertices in H_2 and every neighbour of u in H_2 other than y_1 is super-heavy, by Claim 6.3, there is a super-heavy pair of vertices in G with distance along the cycle C at most two. Hence, G is pancyclic by Lemma 1.4 or 2.5. This final contradiction completes the proof.

□

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